On Classical and Quantum Lattice Spin Systems

by

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Declarations

The work presented in this thesis has been carried out under the supervision of Dr. Daniel Ueltschi.

In Chapters 2, Chapter 3 and in the Appendices are collected some standard results in the field, which are not my own. I have throughout quoted the main sources I have used in the exposition. Parts of this thesis have been published by the author jointly with collaborators:

- Sections 4.2 –4.5 are from [8], a joint publication with Prof. J. Fröhlich and Dr. D. Ueltschi. Apart from minor changes, the text of the paper is reproduced almost verbatim.
- Chapter 5 (except for section 5.3.3) is formed from [9] and [10], respectively an original paper and a review paper in collaboration with Dr. B. Lees and Dr. D. Ueltschi. Parts of these papers are reproduced almost verbatim. The original results stated in Theorem 5.4, Corollary 5.3, Theorem 5.5, Proposition 5.1, Theorem 5.7 and their proofs, which first appeared in our joint publication [9] have been discussed by Dr. B. Lees in his PhD thesis by the title *Quantum spin systems, probabilistic representations and phase transitions* (University of Warwick, 2016).

I declare that the work presented is my own except for what mentioned above and when otherwise stated. This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

This thesis focuses on some results about quantum and classical lattice spin systems.

We study a wide class of two-dimensional quantum models which enjoy a U(1) symmetry. Using the so called complex rotation method we show that the decay of the relevant correlation functions is at least algebraically fast. We provide some examples of relevant models which belong to our class.

We review some results present in the literature concerning the so called Griffiths-Ginibre inequalities for the classical XY model and propose a generalisation to its quantum counterpart. Correlation inequalities indeed hold for the quantum XY model with spin-$\frac{1}{2}$ and for the ground state of the spin-1 system. We propose some applications of these results, namely that the infinite volume limit of some correlation functions exists and that it is possible to compare quenched and annealed averages for a quantum XY model with random couplings.

We investigate loop representations for $\mathrm{O}(n)$ classical spin systems. We propose a generalised random current representation and show its relationship with the Brydges-Fröhlich-Spencer one. We review some conjectures regarding the expected behaviour of these loop models – namely that macroscopic loops appear and their lengths are distributed according to a Poisson-Dirichlet distribution. We propose some arguments in favour of these conjectures for $\mathrm{O}(n)$ loop models, using a mix of exact results and heuristic considerations. In order to do so we define a stochastic process which is an effective split-merge process for macroscopic loops and we reformulate some correlation functions for the $\mathrm{O}(2)$ spin system in terms of loop properties.

Chapter 1

Introduction

Statistical mechanics is the field of mathematical physics concerned with the modelling of the collective behaviour of a large number of particles starting from the individual properties of each particle. An important application is the description of the behaviour of magnetic materials at different temperatures. This involves many approximations of the real system – in particular, particles are supposed to hold fixed positions in space, thus constituting a *lattice*, and to interact only via their magnetic momentum, the so called *spin*. The wide class of models emerging from these approximations takes the collective name of *lattice spin systems*. These mathematical models are highly non trivial and have been the object of intense research for many decades.

There are two classes of lattice spin systems: one might be interested in *quantum* systems, or in the *classical* (i.e. not quantum) setting. Though there are many similarities, it often happens that the behaviour of quantum and classical models differs, and different technical tools are needed to explore their structure. Classical spin systems find their natural description in the language of probability theory, while their quantum counterparts benefit from the tools provided by linear algebra and matrix analysis. In this thesis we discuss some recent results regarding both classical and quantum lattice systems. We provide here a brief overview of the main topics covered, together with a more detailed description of the structure of this work.

The behaviour of statistical mechanical models depends heavily on the temperature at which we study them. A striking a phenomenon is *phase transitions* – we focus in particular on the so called *spontaneous symmetry breaking*. When the temperature is lowered below a critical value, many models exhibit an ordered behaviour, and the individual degrees of freedom tend to show the same collective drive. This phenomenon depends deeply on the dimensionality of the system under study, as stated by the Mermin-Wagner Theorem [56], a cornerstone of the theory of phase transitions – namely, there can not be spontaneous breaking of a continuous symmetry in $d = 1, 2$. We explore some consequences of this result for 2d quantum models in Chapter 4.

A tool which has been useful in the past in the investigation of the behaviour of classical lattice spin systems is correlation inequalities, in particular the so called Griffiths-Ginibre inequalities [34, 32]. While they are well established for many classical spin systems, in the quantum setting this has not been explored thoroughly yet. We discuss correlation inequalities for some quantum models in Chapter 5.

New bridges between statistical mechanics, probability and combinatorics have been recently built thanks to a class of probabilistic models of interacting loops on a lattice which take the name of *random loop models* or *loop soup models* [14, 33, 65, 78]. Indeed, many of them are directly related to some quantum and classical spin systems. In Chapters 6 and 7 we investigate the relationship between loop models and $\mathrm{O}(n)$ lattice systems and explore some conjectures regarding the behaviour of the loops.

This thesis is thus organised as follows.

- In Chapters 2 and 3 we present a brief review of the main tools needed in the study of finite an infinite volume classical and quantum lattice spin models [41, 23, 80].

- The focus of Chapter 4 is on some original results about quantum models on two-dimensional lattices [8]. We present some preliminary remarks on Mermin-Wagner Theorem and its role in the physics and mathematics literature. We briefly review the link between the decay of correlations and the absence of spontaneous symmetry breaking. We then discuss the results from [8] – we identify a wide class of 2d-models with $\mathrm{U}(1)$ symmetry and prove an algebraic bound for the decay of the relevant correlation functions. Various examples of models of interest belonging to this class are provided.

- Chapter 5 focuses on Griffiths inequalities, an incredibly valuable tool in the study of the thermodynamic limit and phase transitions of lattice system. This chapter follows our review paper [10] and our original work [9] closely. We provide an overview on Griffiths inequalities for the classical XY model, collecting and reorganising some results of old [32, 60, 57, 47]. We present our original results from [9] – namely that Griffiths inequalities hold for the quantum XY model with spin-$\frac{1}{2}$ (which has been proved independently in

[31, 76, 67, 9]) and for the ground state of the model with spin-1. Correlation inequalities allow us to compare the critical temperatures for the Ising and the XY model. We also present some new applications – in particular, the infinite volume limit of certain correlation functions is well defined [8], and it is possible to compare quenched and annealed averages of correlations of quantum XY models with random couplings.

- Chapter 6 revolves around the correspondence between certain classical spin systems – the so called $O(n)$ models – and a gas of interacting loops on the same lattice. The results presented here are yet to be published. We review the celebrated Brydges-Fröhlich-Spencer (BFS) representation [14] for the $O(n)$ models. We propose a new generalised random current representation for $O(n)$ models, which takes its inspiration from the random current representation of the Ising model [1]. We explore the relationship between BFS and generalised random current representations and how they can mapped one into the other. We also provide some explicit calculations in order to express certain correlation functions of $O(n)$ spins systems in terms of loop properties.

- Chapter 7 regards some conjectures about loop models. It is expected that a wide class of loop models exhibit macroscopic loops whose lengths should be distributed according to a Poisson-Dirichlet distribution $PD(\vartheta)$ [33, 78]. We focus on $O(n)$ loop models and provide some support for this conjecture. In particular, we propose a stochastic process for the loop model which is an effective split-merge process for long loops, and use it to estimate the value of ϑ for these models. This estimate is confirmed for the $O(2)$ model by a different argument, based on some exact calculations regarding certain correlation functions and some heuristic considerations. These arguments in support of the conjectured $PD(\vartheta)$ behaviour of $O(n)$ loop models have not been published yet.

- Appendices A, B, C and D are devoted to briefly review some well known results which are mentioned and used in the main chapters of this work – namely Trotter formula [80], Hölder inequality for matrices [7], and some features of the random current representation for the Ising model [18] and of Poisson-Dirichlet distributions [78].

Chapter 2

A review of classical lattice systems

This chapter is devoted to a brief introduction to classical lattice spin systems [41, 23]. We first discuss finite volume models without and with boundary conditions, then we define infinite volume systems via the DLR condition. Extremal Gibbs states and extremal states decomposition are also described. We briefly discuss the concept of symmetry and state a version of Mermin Wagner Theorem for a class of finite range interactions.

2.1 Finite volume systems

Let $(\Lambda,\, \mathcal{E})$ be a finite graph, i.e. a finite collection of sites Λ with set of edges $\mathcal{E}$. The most common scenario is given by $\Lambda \subset\subset \mathbb{Z}^d$, and that is what we assume for the rest of the chapter. The notions introduced here can be also extended to more general lattices.

Notation. Given a collection of sites A, we denote by $|A|$ its number of sites. Unless specified otherwise, given $(\Lambda, \mathcal{E})$ we denote by $d(x, y)$ the graph distance between sites $x, y \in \Lambda$, i.e. the length of the shortest path in $(\Lambda, \mathcal{E})$ connecting the two. Analogously, $\mathrm{diam}(A)$ for $A \subset \Lambda$ is the diameter of A calculated according to the graph distance.

The physical idea is that each site $x \in \Lambda$ hosts a physical degree of freedom (spin) – an element σ_x of some compact metric space Ω_0 with Borel σ-algebra $\mathscr{B}_0$ and measure μ_0 over $(\Omega_0, \mathscr{B}_0)$. The set of possible configurations, or configuration space, is then $\Omega_\Lambda = \Omega_0^\Lambda$, which is naturally endowed with the product measure μ_0^Λ. An element $\sigma \in \Omega_\Lambda$ is then $\sigma = \{\sigma_x\}_{x\in\Lambda}$ with $\sigma_x \in \Omega_0$ for any $x \in \Lambda$.

Example 2.1. The simplest case is when on each site there is a discrete spin which takes possible values ± 1, i.e. $\Omega_0 = \{-1, +1\}$. μ_0 is the counting measure. The configuration space is then $\Omega_\Lambda = \{-1, 1\}^\Lambda$. An element of this space is just an assignment of +1 or -1 on each site of the lattice.

Example 2.2. Ω_0 needs not be a discrete space. For a wide class of models $\Omega_0 = \mathbb{S}^n$ for some $n \in \mathbb{N}$. In this case, μ_0 is the usual uniform measure on the sphere $\mathbb{S}^n$.

The degrees of freedom described by Ω_Λ interact through the Hamiltonian, a function $H_\Lambda : \Omega_\Lambda \to \mathbb{R}$ which describes the energy of configurations. It is better formulated in terms of interactions. An interaction $\Phi = \{\Phi_X\}_{X \subset \mathbb{Z}^d}$ is a collection of functions $\Phi_X : \Omega_X \to \mathbb{R}$ for any $X \subset\subset \mathbb{Z}^d$, i.e. each Φ_X depends only on the spins $\{\sigma_x\}_{x \in X}$ while all the others are left unvaried by its action. A common assumption is translation invariance: $\Phi_X = \Phi_{X+\vec{v}}$, where $X+\vec{v}$ is the set obtained by translating X by $\vec{v} \in \mathbb{Z}^d$. To simplify the discussion, we assume translation invariance throughout this chapter. We also assume absolute summability of interaction: $\sum_{X \ni 0} \|\Phi_X\|_\infty < \infty$. The hamiltonian on the volume $\Lambda \subset\subset \mathbb{Z}^d$ is then the sum of local terms:

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi_X. \tag{2.1}$$

An interaction has *finite range* if there exists some $R > 0$ such that $\Phi_X = 0$ if $\operatorname{diam}(X) > R$. Nearest neighbours interactions are those for which $R = 1$, i.e. only sites one next to the other interact.

The *finite volume Gibbs state* at inverse temperature β for a system with configuration space Ω_Λ, measure μ_0^Λ and hamiltonian H_Λ is the linear functional $\langle\cdot\rangle_{\Lambda,\beta}$ that to any function $f : \Omega_\Lambda \to \mathbb{R}$ associates

$$\langle f \rangle_{\Lambda,\beta} = \frac{1}{Z_{\Lambda,\beta}} \int_{\Omega_\Lambda} d\mu_0^\Lambda(\sigma)\, f(\sigma)\, \mathrm{e}^{-\beta H_\Lambda(\sigma)}, \text{ with } Z_{\Lambda,\beta} = \int_{\Omega^\Lambda} d\mu_0^\Lambda(\sigma) \mathrm{e}^{-\beta H_\Lambda(\sigma)}. \tag{2.2}$$

$\langle f \rangle_{\Lambda,\beta}$ is also called *expectation value* of f. $Z_{\Lambda,\beta}$ is called *partition function.*

Analogously $\langle\cdot\rangle_{\Lambda,\beta}$ can be seen as the average with respect to the measure $\mu_{\Lambda,\beta}$ on Ω_Λ such that

$$d\mu_{\Lambda,\beta}(\sigma) = d\mu_0^\Lambda \frac{1}{Z_{\Lambda,\beta}} \mathrm{e}^{-\beta H_\Lambda(\sigma)}. \tag{2.3}$$

This is called *finite volume Gibbs measure.*

Example 2.3 (The Ising model)**.** A typical nearest neighbour model is the Ising ferromagnet. For this system $\Omega_\Lambda = \{-1, 1\}^\Lambda$ and μ_0 is the counting measure. The hamiltonian is

$$H_\Lambda(\sigma) = -\frac{1}{2} \sum_{x,y\in\Lambda} J_{xy}\, \sigma_x \sigma_y, \tag{2.4}$$

with

$$J_{xy} \begin{cases} \geq 0 & \text{if } d(x,y) = 1, \\ = 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

In the following examples we focus on the case $J_{xy} = J \geq 0$ for any pair of nearest neighbours x and y. The expectation value of $f : \Omega_\Lambda \to \mathbb{R}$ is

$$\begin{aligned} \langle f\rangle_{\Lambda,\beta} &= \frac{1}{Z_{\Lambda,\beta}} \sum_{\sigma\in\{-1,1\}^\Lambda} f(\sigma)\, \mathrm{e}^{\frac{\beta}{2}\sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y}, \quad \text{with} \\ Z_{\Lambda,\beta} &= \sum_{\sigma\in\{-1,1\}^\Lambda} \mathrm{e}^{\frac{\beta}{2}\sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y}. \end{aligned} \tag{2.6}$$

The Ising model is one of the most studied models in statistical mechanics, proposed by Lenz [51], and solved in the one-dimensional case by his PhD student Ising [40]. The more challenging two-dimensional case was solved later by Onsager [64].

Example 2.4 (O(n) models)**.** Another class of well studied models is given by O(n) models. In their standard formulation they are nearest neighbours models. In this case $\Omega_0 = \mathbb{S}^{n-1}$ for some $n \in \mathbb{N}$. A configuration $\sigma \in \Omega_\Lambda$ is then given by $\sigma = \{\vec{\sigma}_x\}_{x\in\Lambda}$ with $\vec{\sigma}_x \in \mathbb{S}^{n-1}$ for any $x \in \Lambda$. The hamiltonian is defined as

$$H_\Lambda(\sigma) = -\frac{1}{2} \sum_{x,y\in\Lambda} J_{xy}\, \vec{\sigma}_x \cdot \vec{\sigma}_y. \tag{2.7}$$

with J_{xy} as in eq. (2.5). We focus on the case $J_{xy} = J_{yx} = J \geq 0$ for any pair of nearest neighbours x and y. The expectation value of $f : \Omega_\Lambda \to \mathbb{R}$ is

$$\begin{aligned} \langle f\rangle_{\Lambda,\beta} &= \frac{1}{Z_{\Lambda,\beta}} \int_{(\mathbb{S}^{n-1})^\Lambda} \prod_{x\in\Lambda} d\vec{\sigma}_x\, f(\sigma) \mathrm{e}^{\frac{\beta}{2}\sum_{x,y\in\Lambda} J_{xy}\, \vec{\sigma}_x\cdot\vec{\sigma}_y}, \quad \text{with} \\ Z_{\Lambda,\beta} &= \int_{(\mathbb{S}^{n-1})^\Lambda} \prod_{x\in\Lambda} d\vec{\sigma}_x\, \mathrm{e}^{\frac{\beta}{2}\sum_{x,y\in\Lambda} J_{xy}\, \vec{\sigma}_x\cdot\vec{\sigma}_y}, \end{aligned} \tag{2.8}$$

and $d\vec{\sigma}_x$ being the uniform measure on the $(n-1)$-sphere at site x. Notice that the case $n = 1$ is the Ising model. The model with $n = 2$ is known as XY or rotor model, and the case $n = 3$ is the so called Heisenberg model.

2.1.1 Boundary conditions

So far we have focused on the situation where Λ is completely isolated and effectively there is no "outside" of this finite volume. Nonetheless, one can fix a configuration outside Λ, the so called external configuration, and let the degrees of freedom inside the volume interact with this fixed choice of configuration in Λ^c. A choice of external configuration is called boundary condition.

Define $\Omega = \Omega_0^{\mathbb{Z}^d}$ and $\Omega_{\Lambda^c} = \Omega_0^{\Lambda^c}$. Let $\tau \in \Omega_{\Lambda^c}$. The finite volume hamiltonian with boundary condition τ, $H_\Lambda^\tau : \Omega_\Lambda \to \mathbb{R}$ is

$$H_\Lambda^\tau(\sigma) = \sum_{\substack{X \subset\subset \mathbb{Z}^d: \\ X\cap\Lambda\neq\emptyset}} \Phi_X(\sigma\circ\tau), \tag{2.9}$$

where $\sigma\circ\tau \in \Omega$ is the configuration composed by σ inside the volume Λ and τ outside of it. Notice that H_Λ^τ is still a function on Ω_Λ since τ is fixed.

Example 2.5. For the Ising model, let $\tau \in \{-1,1\}^{\Lambda^c}$. The hamiltonian with boundary condition τ is then

$$H_\Lambda^\tau(\sigma) = -\frac{1}{2}\sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y - \sum_{\substack{x\in\Lambda, \\ y\in\Lambda^c}} J_{xy}\sigma_x\tau_y \tag{2.10}$$

with J_{xy} as in Eq. 2.5. Notice that only the sites at the boundary of Λ interact with the outside and H_Λ^τ is still the sum of a finite number of terms. There are two boundary conditions that are usually considered: $\tau_x = 1\,\forall x\in\Lambda^c$ and $\tau_x = -1\,\forall x \in \Lambda^c$.

Example 2.6. For the O(n) model, let $\tau \in (\mathbb{S}^{(n-1)})^{\Lambda^c}$. The hamiltonian with boundary condition τ is then

$$H_\Lambda^\tau(\sigma) = -\frac{1}{2}\sum_{x,y\in\Lambda} J_{xy}\,\vec{\sigma}_x\cdot\vec{\sigma}_y - \sum_{\substack{x\in\Lambda, \\ y\in\Lambda^c}} J_{xy}\,\vec{\sigma}_x\cdot\vec{\tau}_y, \tag{2.11}$$

with J_{xy} as in Eq. (2.5). Notice that also in this case only the sites at the boundary interact with the outside, and H_Λ^τ is given by the sum of a finite number of terms. The usual boundary conditions are $\vec{\tau}_x = \vec{\tau}$ with $\vec{\tau}\in\mathbb{S}^{(n-1)}$ for all $x\in\Lambda^c$.

In general we denote $\langle\cdot\rangle_{\Lambda,\beta}^\tau$ and $\mu_{\Lambda,\beta}^\tau$ the Gibbs state and measure at inverse temperature β with boundary condition τ. It is the same as Eq.s (2.2), (2.3) but for the fact that we use H_Λ^τ instead of H_Λ. The Gibbs state and measure described in Eq.s (2.2), (2.3) are usually said to have *open* or *free* boundary conditions.

2.2 Infinite volume systems

Interesting behaviours of statistical mechanical models are usually better studied in the infinite volume limit $\Lambda \nearrow \mathbb{Z}^d$ – this limit naturally provides a description of the bulk properties of the model. Defining it properly is a delicate problem. In order to avoid the difficulties emerging from a direct approach to its definition, we take a different perspective, and define infinite volume Gibbs states through the so called DLR condition. It takes its name from Dobrušin [16], Lanford and Ruelle [50]. We adopt here the formulation from [23]. Define $\mathfrak{L}_\Omega$ the set of local functions on Ω i.e. $\mathfrak{L}_\Omega = \{f : \Omega \to \mathbb{R} \text{ s.t. } f \text{ depends only on a finite number of spins}\}$.

Definition 2.1 (Infinite volume Gibbs measures)**.** *A measure μ_β on Ω is an infinite volume Gibbs measure at inverse temperature β for the system with interaction Φ if for any $\Lambda \subset\subset \mathbb{Z}^d$*

$$\mu_\beta(f) = \int_\Omega \mu^\tau_{\Lambda,\beta}(f)\mu_\beta(d\tau)$$

for any local function f depending only on spins in Λ (i.e. there exists $\bar{f} : \Omega_\Lambda \to \mathbb{R}$ such that $\bar{f}(\bar{\sigma}) = f(\sigma)$ for any $\bar{\sigma} \in \Omega_\Lambda$, $\sigma \in \Omega$ with $\bar{\sigma}_x = \sigma_x$ for all $x \in \Lambda$). Here $\mu^\tau_{\Lambda,\beta}(\cdot)$ is the finite volume Gibbs state with boundary condition τ for the system with interaction Φ at inverse temperature β.

Given an infinite volume Gibbs measure μ_β, the normalised linear functional $\langle\cdot\rangle_\beta : \mathfrak{L}_\Omega \to \mathbb{R}$ such that $\langle f\rangle_\beta$ is the average of f with respect to μ_β is called an *infinite volume Gibbs state*. In this chapter we always assume to work with translation invariant states, i.e. $\langle f\rangle_\beta = \langle\theta_{\vec{v}} f\rangle_\beta$, where θ_v is a translation by a vector $\vec{v} \in \mathbb{Z}^d$. We denote by $\mathcal{G}_\beta(\Phi)$ the set of infinite volume Gibbs states related to the interaction $\{\Phi_X\}_{X\subset\subset\mathbb{Z}^d}$ at inverse temperature β. It can be proved that $\mathcal{G}_\beta(\Phi)$ is a not empty and convex set – see for example [23], Theorems 6.26 and 6.56.

The most interesting phenomenon in statistical mechanics, phase transitions, has a natural definition in terms of presence of multiple distinct Gibbs states.

Definition 2.2 (Phase transition)**.** *If $|\mathcal{G}_\beta(\Phi)| > 1$ the system with interaction Φ undergoes a phase transition at inverse temperature β.*

Convexity of $\mathcal{G}_\beta(\Phi)$ allow us to define extremal states as follows [23].

Definition 2.3 (Extremal Gibbs states)**.** *Let $\langle\cdot\rangle_\beta \in \mathcal{G}_\beta(\Phi)$. It is called extremal if any convex decomposition of the form $\langle\cdot\rangle_\beta = t\langle\cdot\rangle'_\beta + (1-t)\langle\cdot\rangle''_\beta$ with $t \in (0,1)$ and $\langle\cdot\rangle'_\beta$, $\langle\cdot\rangle''_\beta \in \mathcal{G}_\beta(\Phi)$ implies $\langle\cdot\rangle'_\beta = \langle\cdot\rangle''_\beta = \langle\cdot\rangle_\beta$.*

The set of extremal states is denoted by $\mathrm{ex}\mathcal{G}_\beta(\Phi)$. The measures related to extremal states are called extremal measures.

Example 2.7 (Extremal states for the Ising model)**.** Let $\langle\cdot\rangle^{+}_{\Lambda,\beta}$ and $\langle\cdot\rangle^{-}_{\Lambda,\beta}$ denote the Gibbs state for the Ising model on the finite volume Λ with boundary condition $\tau_x = \pm 1 \; \forall x \in \Lambda^c$. It can be proved that $\lim_{\Lambda \nearrow \mathbb{Z}^d} \langle\cdot\rangle^{\pm}_{\Lambda,\beta}$ exists, where the limit is taken along any increasing sequence of volumes (e.g. along a sequence of concentric boxes of increasing diameter) – see [23], Theorem 3.17 – and fulfil the DLR condition. We denote these two infinite volume Gibbs states $\langle\cdot\rangle^{\pm}_{\beta}$, and the infinite volume Gibbs measures $\mu^{\pm}_{\beta}$. $\langle\cdot\rangle^{\pm}_{\beta}$ are the two translation invariant extremal states for the Ising model of Example 2.3, see [23], Lemma 6.65.

Example 2.8 (Extremal states for the O(2) model)**.** Let us consider the $\mathrm{O}(n)$ model of Ex. 2.4 with $n = 2$, i.e. each spin lies on the circumference $\mathbb{S}^1$. We denote by $\langle\cdot\rangle^{\vec{\tau}}_{\beta}$ the infinite volume Gibbs states obtained as $\lim_{\Lambda \nearrow \mathbb{Z}^d} \langle\cdot\rangle^{\vec{\tau}}_{\Lambda,\beta}$, where $\langle\cdot\rangle^{\vec{\tau}}_{\Lambda,\beta}$ has uniform boundary condition $\vec{\tau}_x = \vec{\tau}$ for any $x \in \Lambda^c$ for some $\vec{\tau} \in \mathbb{S}^1$. It has been proved that they form the set of translation invariant extremal states, $\mathrm{ex}\mathcal{G}_\beta = \{\langle\cdot\rangle^{\vec{\tau}}_{\beta} : \vec{\tau} \in \mathbb{S}^1\}$ [27].

Notation. Another common notation for the extremal states for the O(2) model is as follows. Any $\vec{\tau} \in \mathbb{S}^1$ can be expressed in polar coordinates, $\vec{\tau} = (\cos\alpha, \sin\alpha)$ for some $\alpha \in [0, 2\pi)$. Then the extremal Gibbs state $\langle\cdot\rangle^{\vec{\tau}}_{\beta}$ can be denoted also by $\langle\cdot\rangle^{\alpha}_{\beta}$.

Extremal states constitute the "building blocks" for all other states. This is expressed by the following theorem.

Theorem 2.1 (Extremal states decomposition)**.** *For any $\langle\cdot\rangle_\beta \in \mathcal{G}_\beta(\Phi)$ there exists a unique measure ν over $\mathcal{G}_\beta(\Phi)$ concentrated on ex$\mathcal{G}_\beta(\Phi)$ such that $\langle\cdot\rangle_\beta$ is the barycentre of ν.*

This theorem relies on Choquet's Theorem and on properties of Choquet simplexes – see [41] for the proof of this statement.

Remark. If the set of extremal states is endowed in a canonical way with a suitable σ-algebra, Theorem 2.1 can be formulated as follows. To simplify the notation, let us label extremal states by some index in some suitable set $A_{\Phi,\beta}$ endowed with a suitable σ-algebra $\mathscr{B}_{\Phi,\beta}$ so that $\langle\cdot\rangle^{\alpha}_{\beta} \in \mathrm{ex}\mathcal{G}_\beta(\Phi)$ with $\alpha \in A_{\Phi,\beta}$ (i.e. there is a bijection between ex$\mathcal{G}_\beta(\Phi)$ and $A_{\Phi,\beta}$). Then Theorem 2.1 amounts to saying that for any $\langle\cdot\rangle_\beta \in \mathcal{G}_\beta(\Phi)$ there exists a measure ν over $(A_{\Phi,\beta}, \mathscr{B}_{\Phi,\beta})$ such that for any $f \in \mathfrak{L}_\Omega$

$$\langle f\rangle_\beta = \int_{A_{\Phi,\beta}} d\nu\,(\alpha)\,\langle f\rangle^{\alpha}_{\beta}. \tag{2.12}$$

See [41] Theorem IV.3.3 or [23] Theorem 6.87 for a proof of this statement.

Example 2.9 (Extremal states decomposition for the Ising model with open boundary conditions)**.** Let us consider the Ising model discussed previously and denote by $\langle\cdot\rangle_\beta$ the infinite volume Gibbs state obtained by $\lim_{\Lambda\nearrow\mathbb{Z}^d}\langle\cdot\rangle_{\Lambda,\beta}$ – $\langle\cdot\rangle_{\Lambda,\beta}$ being the finite volume Gibbs state with open boundary conditions. By the results discussed in Ex. 2.7 and by the $\mathbb{Z}_2$ symmetry of the model (see Example 2.11) of the model, for any $f \in \mathfrak{L}_\Omega$

$$\langle f\rangle_\beta = \frac{1}{2}\left(\langle f\rangle_\beta^+ + \langle f\rangle_\beta^-\right), \tag{2.13}$$

i.e. ν in Theorem 2.1 is the uniform measure on extremal states.

Example 2.10 (Extremal states decomposition for the O(2) model with open boundary condition)**.** Let us consider the O(2) model discussed previously and denote by $\langle\cdot\rangle_\beta$ the infinite volume Gibbs state obtained by $\lim_{\Lambda\nearrow\mathbb{Z}^d}\langle\cdot\rangle_{\Lambda,\beta}$ – $\langle\cdot\rangle_{\Lambda,\beta}$ being the finite volume Gibbs state with open boundary conditions. By the results described in Ex. 2.8 and by the O(2) symmetry of the model, we have that for any $f \in \mathfrak{L}_\Omega$

$$\langle f\rangle_\beta = \int_0^{2\pi}\frac{d\alpha}{2\pi}\langle f\rangle_\beta^\alpha, \tag{2.14}$$

i.e. the measure ν in Theorem 2.1 is the uniform measure on extremal states.

Remark. Recall that extremal states might coincide. Indeed, if there is no phase transition, all the states coincide. The concept of extremal Gibbs state is then of particular relevance for those β such that a phase transition takes place, and $|\mathcal{G}_\beta(\Phi)| > 1$.

We conclude this section by stating that extremal states enjoy some *cluster properties*, as proved by Ruelle and Lanford [50] (Propositions 2.3–2.4 and Theorem 3.4) and discussed in Israel's book [41] (Lemma IV.3.9).

Theorem 2.2 (Cluster properties of extremal states)**.** *Let $\langle\cdot\rangle_\beta \in ex\mathcal{G}_\beta(\Phi)$. Then for any f, g local functions*

$$\lim_{\|x\|\to\infty}\langle f\,\theta_x g\rangle_\beta = \langle f\rangle_\beta\langle g\rangle_\beta$$

with θ_x the translation by a vector x.

Physically, this theorem tells us that for extremal states local functions whose supports are far apart are essentially uncorrelated.

2.3 Symmetries and Mermin-Wagner Theorem

The concept of symmetry has a role of primary importance in the theory of phase transitions. It can be formally defined as follows.

Definition 2.4 (Symmetry)**.** *Let $\Phi = \{\Phi_X\}_{X \subset\subset \mathbb{Z}^d}$ be an interaction on the state space Ω. A symmetry for this model is a group G which acts on Ω_0 via a group of transformations $\{\tau_g\}_{g\in G}$, $\tau_g : \Omega_0 \to \Omega_0$ such that μ_0 is G-invariant and*

$$\Phi_X(\tau_g\sigma) = \Phi_X(\sigma)$$

for any $g \in G$, any $X \subset\subset \mathbb{Z}^d$ and any $\sigma \in \Omega$, where $\tau_g\sigma = \{\tau_g\sigma_x\}_{x\in\mathbb{Z}^d}$.

Example 2.11. Notice that the Ising model is $\mathbb{Z}_2$-symmetric. Indeed, we have that $\Phi_X(\sigma) = \Phi_X(-\sigma)$ for any $\sigma \in \Omega$ and $X \subset\subset \mathbb{Z}^d$, where Φ is the interaction described in Ex. 2.3.

Example 2.12. The $\mathrm{O}(n)$ models owe their name to the fact that they are $\mathrm{O}(n)$ invariant. Indeed, given $R : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ a rotation on the $(n-1)$-sphere and given $\sigma \in \Omega$, define $R\sigma = \{R\sigma_x\}_{x\in\Omega}$. Then

$$\Phi_X(\sigma) = \Phi_X(R\sigma) \tag{2.15}$$

for any $\sigma \in \Omega$ and any $X \subset\subset \mathbb{Z}^d$, where the interaction Φ is the one described in Ex. 2.4. Moreover, the uniform measure over $\mathbb{S}^{n-1}$ is clearly $\mathrm{O}(n)$ invariant.

Notice that $\mathcal{G}_\beta(\Phi)$ is preserved by the action of the group G, as stated here and proved e.g. in [23].

Proposition 2.1. *Given $\langle\cdot\rangle_\beta \in \mathcal{G}_\beta(\Phi)$ and $g \in G$ define $\langle\cdot\rangle_\beta^g : \mathfrak{L}_\Omega \to \mathbb{R}$ such that $\langle f\rangle_\beta^g = \langle f \circ \tau_g\rangle_\beta$ for any $f \in \mathfrak{L}_\Omega$. Then $\langle\cdot\rangle_\beta^g \in \mathcal{G}_\beta(\Phi)$ for any $g \in G$.*

Remark. This statement has an interesting consequence. If there is no phase transition at inverse temperature β, i.e. $|\mathcal{G}_\beta(\Phi)| = 1$ and $\mathcal{G}_\beta(\Phi) = \{\langle\cdot\rangle_\beta\}$, then $\langle\cdot\rangle_\beta = \langle\cdot\rangle_\beta^g$ for any $g \in G$.

Example 2.13. Consider the Ising model defined in Ex. 2.3. From Ex. 2.11 we know that it is $\mathbb{Z}_2$ invariant. We have seen in Ex. 2.7 that the translation invariant extremal states are $\langle\cdot\rangle_\beta^\pm$, defined as infinite volume limit of the finite volume Gibbs states $\langle\cdot\rangle_{\Lambda,\beta}^\pm$. Notice that each of them can be obtained applying a $\mathbb{Z}_2$ transformation to the other. Indeed, for any local function $f : \Omega \to \mathbb{R}$, define f^- as $f^-(\sigma) = f(-\sigma)$ $\forall\sigma \in \Omega$. Then it is straightforward to check that

$$\langle f\rangle_\beta^- = \langle f^-\rangle_\beta^+. \tag{2.16}$$

We introduce now the concept of spontaneous symmetry breaking, which is closely related to the idea of phase transition.

Definition 2.5 (Spontaneous symmetry breaking)**.** *Let G be a symmetry for the model with interaction Φ and state space Ω. If there exists $\langle\cdot\rangle_\beta \in \mathcal{G}_\beta(\Phi)$ such that*

$$\langle\cdot\rangle_\beta \neq \langle\cdot\rangle_\beta^g$$

the system undergoes a spontaneous symmetry breaking at inverse temperature β.

Notice that due to Proposition 2.1 and to the Remark above, spontaneous symmetry breaking is a particular instance of phase transition – indeed, for spontaneous symmetry breaking to take place, it is necessary that $|\mathcal{G}_\beta(\Phi)| > 1$.

Example 2.14. Let us consider the Ising model. For any finite volume, it is clear that $\langle\cdot\rangle_{\Lambda,\beta}^{\pm}$ explicitly breaks the $\mathbb{Z}_2$ symmetry. Nonetheless, phase transitions and spontaneous symmetry breaking are well defined only in the infinite volume scenario. Indeed, it is a very well known result that on the hypercubic lattice with $d \geq 2$, the Ising model undergoes a phase transition with spontaneous symmetry breaking. More precisely, there exists $\beta_c < \infty$ such that if $\beta > \beta_c$ (i.e. at low temperature) we have multiple Gibbs states, in particular $\langle\cdot\rangle_\beta^+ \neq \langle\cdot\rangle_\beta^-$ (i.e. there is spontaneous symmetry breaking). On the other hand, if $\beta < \beta_c$ (i.e. at high temperature) all Gibbs states coincide. See for example [23], Chapter 3, for a recent review.

We mention now a celebrated result known by the name of Mermin-Wagner Theorem [56]. Heuristically, this theorem states that there can not be spontaneous symmetry breaking of a continuous symmetry in 1d and 2d systems. This sort of result was investigated first in the quantum setting and later in the classical one. The first work in this direction was a seminal paper by Mermin and Wagner [56] about the absence of spontaneous magnetisation in the quantum Heisenberg model. We quote here the recent formulation for two-dimensional systems from [39]. It is a generalisation of previous results [17].

Theorem 2.3. *Let $\Phi = \{\Phi_X\}_{X\subset\subset\mathbb{Z}^2}$ be a translation invariant interaction with finite range. Assume that Φ_X is continuous and bounded for any $X \subset\subset \mathbb{Z}^2$. Let G be a compact connected Lie group and assume the model is G-symmetric. Then for any $\beta \in [0,\infty)$ and any $\langle\cdot\rangle_\beta \in \mathcal{G}_\beta(\Phi)$*

$$\langle\cdot\rangle_\beta = \langle\cdot\rangle_\beta^g \quad \forall g \in G.$$

See [39] for the proof. This type of statement can be generalised also for infinite range interactions as long as they decay fast enough, see [68, 39].

Remark. Notice that we have seen in Ex. 2.14 that the Ising model undergoes spontaneous symmetry breaking for finite β also in $d = 2$. This is due to the fact that the symmetry group of the Ising model ($\mathbb{Z}_2$) is discrete, so Theorem 2.3 does not apply.

Chapter 3

A review of quantum lattice systems

In this chapter we provide a brief overview of quantum lattice spin systems [41, 80]. We discuss finite volume systems, and we use the KMS condition to define infinite volume ones. Extremal KMS states and extremal states decomposition are described. We conclude by defining the concept of symmetry and briefly discussing Mermin-Wagner Theorem.

3.1 Finite volume systems

As in the previous chapter, let Λ be a finite collection of sites, and let $\mathcal{E}$ be the set of its edges. For simplicity, we assume $\Lambda \subset\subset \mathbb{Z}^d$. The possible physical states are described by vectors belonging to a Hilbert space $\mathcal{H}_\Lambda$, which plays a similar role to Ω_Λ in the classical setting. Here we focus on Hilbert spaces with finite dimensions.

Notation. Given $\phi \in \mathcal{H}_\Lambda$, it is very common to find it denoted as $|\phi\rangle$. Its conjugate ϕ^* is denoted as $\langle\phi|$. With this notation the scalar product (ϕ, ψ) is written as $\langle\phi|\psi\rangle$. This convention is known as Dirac notation. $\langle\cdot|$ is called *bra* and $|\cdot\rangle$ is called *ket*.

The interaction between the quantum degrees of freedom described by $\mathcal{H}_\Lambda$ is formulated by linear operators acting on $\mathcal{H}_\Lambda$. The algebra of such operators is denoted by $\mathcal{B}(\mathcal{H}_\Lambda)$. Notice that $\mathcal{B}(\mathcal{H}_\Lambda)$ with the usual conjugation and operator norm is a C^*-algebra, i.e. it has the following properties.

Definition 3.1. *A C^*-algebra $\mathcal{C}$ is an associative normed algebra such that:*

- *There exists a conjugation map $* : \mathcal{C} \to \mathcal{C}$ such that for all $a, b \in \mathcal{C}$ and*

$\alpha, \beta \in \mathbb{C}$:

$$(ab)^* = b^* a^*$$
$$(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$$

i.e. $\mathcal{C}$ is a $*$-algebra.

- *The norm defined on $\mathcal{C}$ respects the so called C^*-property: for all $A \in \mathcal{C}$*

$$\|a^* a\| = \|a\|^2.$$

Operators in $\mathcal{B}(\mathcal{H}_\Lambda)$ are often called *observables*.

Example 3.1. The Hilbert space of a single particle of spin s is $\mathbb{C}^{2s+1}$. Some particularly relevant operators in $\mathcal{B}(\mathbb{C}^{2s+1})$ are the three *spin matrices*. These operators $\{\mathcal{S}^i\}_{i=1}^3$ are hermitian and satisfy the following relations:

$$\left[\mathcal{S}^1, \mathcal{S}^2\right] = i\mathcal{S}^3 \text{ and its cyclic permutations,}$$
$$(\mathcal{S}^1)^2 + (\mathcal{S}^2)^2 + (\mathcal{S}^3)^2 = s(s+1)\mathbb{1}_{\mathbb{C}^{2s+1}}.$$

We have introduced the notation $[a, b] = ab - ba$. It can be shown that all the three spin matrices have eigenvalues $-s$, $-s+1$, $\dots$, $s-1$, s (see e.g. [80] Lemma 3.2). It is customary to choose $\mathcal{S}^3$ diagonal, and to use its eigenvectors as working basis of $\mathbb{C}^{2s+1}$. These eigenvectors are labelled by eigenvalue: $\{|{-s}\rangle, |{-s+1}\rangle, \dots |s-1\rangle, |s\rangle\}$. In the case of spin-$\frac{1}{2}$ the spin matrices can be formulated through the celebrated Pauli matrices $\{\tau_i\}_{i=1}^3$ i.e. $\mathcal{S}^i = \frac{1}{2}\tau^i$ with:

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In many cases $\mathcal{H}_\Lambda$ can be described as a "composition" of Hilbert spaces related to the single sites. This composition is given by tensor product.

Definition 3.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces of dimension $d_\mathcal{H}$ and $d_\mathcal{K}$ respectively and respective basis $\{e_i\}_{i=1}^{d_\mathcal{H}}$ and $\{f_j\}_{j=1}^{d_\mathcal{K}}$. The Hilbert space $\mathcal{H} \otimes \mathcal{K}$ is a Hilbert space of dimension $d_\mathcal{H} d_\mathcal{K}$ spanned by pairs $\{(e_i, f_j)\}_{i,j}$. The vectors of this basis are usually denoted as $\{e_i \otimes f_j\}_{i,j}$.*

Notice that $\mathcal{H} \otimes \mathcal{K}$ naturally inherits a scalar product from $\mathcal{H}$ and $\mathcal{K}$. Given $\phi_1, \phi_2 \in \mathcal{H}$ and $\psi_1, \psi_2 \in \mathcal{K}$, $(\phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2)_{\mathcal{H}\otimes\mathcal{K}} = (\phi_1, \phi_2)_\mathcal{H}(\psi_1, \psi_2)_\mathcal{K}$, where the subscript denotes on which Hilbert space the scalar product is defined. Moreover, the tensor product naturally extends to operators acting on $\mathcal{H}$ and $\mathcal{K}$: let $a_{\mathcal{H}}$ and $b_{\mathcal{K}}$ be such operators, and $\phi \in \mathcal{H}$, $\psi \in \mathcal{K}$, then $(a_{\mathcal{H}} \otimes b_{\mathcal{K}})(\phi \otimes \psi) = (a_{\mathcal{H}}\phi) \otimes (b_{\mathcal{K}}\psi)$.

Remark. It is very often the case that $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x$, where $\mathcal{H}_x$ is some finite Hilbert space describing the state of a quantum degree of freedom at site x. For simplicity, we are going to assume it throughout the chapter. In this case, it is straightforward to define subalgebras of operators $\{\mathcal{B}_A\}_{A\subset\Lambda}$, $\mathcal{B}_A \subset \mathcal{B}(\mathcal{H}_\Lambda)$ for any $A \subset \Lambda$. Given $A \subset \Lambda$ and defined $\mathcal{H}_A = \otimes_{x\in A}\mathcal{H}_x$, then $\mathcal{B}_A = \{a \otimes \mathbb{1}_{\Lambda\setminus A} : a \in \mathcal{B}(\mathcal{H}_A)\}$.

Example 3.2. The Hilbert space of a model with one particle of spin s on each site of the lattice is $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2s+1}$. The spin operators acting only on a certain site $x \in \Lambda$ are denoted by $\mathcal{S}^i_x = \mathcal{S}^i \otimes \mathbb{1}_{\Lambda\setminus\{x\}}$.

The behaviour of the quantum degrees of freedom on the lattice Λ is described by a hamiltonian H_Λ, which is a hermitian operator acting on $\mathcal{H}_\Lambda$. From a physical point of view, its eigenvalues are the energies of the eigenvectors. The hamiltonian of a system is usually formulated by means of an interaction Φ.

Definition 3.3. *An interaction Φ for a finite volume system is a collection of operators labelled by subsets of Λ, $\Phi = \{\Phi_A\}_{A\subset\Lambda}$ such that for all $A \subset \Lambda$, $\Phi_A \in \mathcal{B}_A$, $\Phi^*_A = \Phi_A$.*

Each operator Φ_A describes a *local interaction* on $A \subset \Lambda$. Given an interaction, the related hamiltonian is formulated as

$$H_\Lambda = \sum_{A\subset\Lambda} \Phi_A.$$

For the rest of the chapter we assume translation invariance, i.e. for any $X \subset\subset \Lambda$ $\Phi_{X+\vec{v}} = \theta_{\vec{v}}\Phi_X$, where $\theta_{\vec{v}}$ is the translation by a vector $\vec{v} \in \mathbb{Z}^d$ i.e. $\theta_{\vec{v}}\mathcal{B}_A = \mathcal{B}_{A+\vec{v}}$. An interaction has *finite range* if there exists some $R > 0$ such that $\|\Phi_A\| = 0$ if $\mathrm{diam}(A) > R$. Nearest neighbours interactions are those for which $R = 1$, i.e. only sites one next to the other interact.

Example 3.3. The hamiltonian of the anisotropic Heisenberg model on Λ is:

$$H_\Lambda = -\sum_{(x,y)\in\mathcal{E}} J^1_{xy}\mathcal{S}^1_x\mathcal{S}^1_y + J^2_{xy}\mathcal{S}^2_x\mathcal{S}^2_y + J^3_{xy}\mathcal{S}^3_x\mathcal{S}^3_y.$$

Here $J^i_{xy} \in \mathbb{R}$ for all $i = 1, 2, 3$ and $(x, y) \in \mathcal{E}$. In case $J^i_{xy} = J$ for all $i = 1, 2, 3$ and $(x, y) \in \mathcal{E}$ this is simply called ferromagnetic (if $J > 0$) or antiferromagnetic (if

$J < 0$) Heisenberg model. If $J^3_{xy} = 0$ and $J^1_{xy}, J^2_{xy} \geq 0$ this is the ferromagnetic XY model – we focus on it in Chapter 5. If $J^1_{xy} = J^2_{xy} \neq J^3_{xy}$ this model takes the name of XXZ model. This is evidently a nearest neighbours interaction.

Given a hamiltonian describing the physical behaviour of our model we need to define *expectation values* of observables i.e. we need some notion of "average". This is provided by states.

Definition 3.4. *A state $\langle\cdot\rangle$ is a linear functional on $\mathcal{B}(\mathcal{H}_\Lambda)$, $\langle\cdot\rangle : \mathcal{B}(\mathcal{H}_\Lambda) \to \mathbb{C}$ with the following properties:*

- *It is normalised: $\langle \mathbb{1} \rangle = 1$.*
- *It is positive: $\langle a^* a\rangle \geq 0$ for all $a \in \mathcal{B}(\mathcal{H}_\Lambda)$.*

Straightforward corollaries of this definition are that expectation values of hermitian operators are always real and that $\langle a \rangle = \overline{\langle a^* \rangle}$ for any $a \in \mathcal{B}(\mathcal{H}_\Lambda)$. The states that are of interest in our physical setting are the so called *Gibbs states*:

$$\langle a \rangle_{\Lambda,\beta} = \frac{\mathrm{Tr}\ a\mathrm{e}^{-\beta H_\Lambda}}{Z_{\Lambda,\beta}}, \qquad Z_{\Lambda,\beta} = \mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda}. \tag{3.1}$$

Above, a is any operator in $\mathcal{B}(\mathcal{H}_\Lambda)$, β is the *inverse of the temperature* of the system and H_Λ is the hamiltonian of our model. The normalisation factor $Z_{\Lambda,\beta}$ is the *partition function.* Notice that Gibbs states in the quantum setting and in the classical one (see Eq. (2.2)) are remarkably similar. Nonetheless, in the quantum case it is not possible to introduce a notion of Gibbs measure, differently from what happens in the classical scenario.

3.2 Time evolution and infinite volume systems

The definition of infinite volume Gibbs states in the quantum setting is a matter even more delicate than in the case of classical models, since it is not immediately evident how to define boundary conditions for quantum systems. We take the approach of defining infinite volume Gibbs states by the so called KMS condition, which plays in the quantum setting the same role of the DLR condition of Def. 2.1 in the classical one. The KMS condition provides us with some constraints on the time evolution of the system, which we discuss in the following.

Definition 3.5 (Finite volume dynamics)**.** *Let us consider a quantum system on the lattice $(\Lambda, \mathcal{E})$ with Hilbert space $\mathcal{H}_\Lambda$ and hamiltonian H_Λ. The time evolution is* *defined as the following one-parameter group of *-automorphisms on* $\mathcal{B}(\mathcal{H}_\Lambda)$*:* α_t^Λ*,* $t \in \mathbb{R}$ *s.t.*

$$\alpha_t^\Lambda(a) = e^{itH_\Lambda} a e^{-itH_\Lambda} \quad \forall\, a \in \mathcal{B}(\mathcal{H}_\Lambda).$$

At finite volume the dynamics functional α_t^Λ can be generalised to $t \in \mathbb{C}$. Finite volume Gibbs states are invariant under time evolution – this is trivially seen by cyclicity of the trace. Notice that for any $a, b \in \mathcal{B}(\mathcal{H}_\Lambda)$ the following equality holds – again, by cyclicity of the trace:

$$\langle a\, \alpha_t^\Lambda(b)\rangle_{\Lambda,\beta} = \langle \alpha_{t-i\beta}^\Lambda(b)\, a\rangle_{\Lambda,\beta}. \tag{3.2}$$

The dynamics is well defined in the infinite volume limit. In order to discuss this result, it is necessary to introduce some notation for the infinite volume setting – see [41, 62, 80].

Definition 3.6. *The set of quasi-local observables* $\mathcal{B}$ *is defined as the norm-completion of the union of the algebras of local observables:*

$$\mathcal{B} = \overline{\bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{B}(\mathcal{H}_\Lambda)}.$$

Since we are trying to have a well defined infinite volume setting, interactions can be defined more generally on the whole of $\mathbb{Z}^d$ – of course the case of finite volume described in Def. 3.3 is a straightforward restriction of this general case. So $\Phi = \{\Phi_A\}_{A \subset\subset \mathbb{Z}^d}$ is a family of hermitian operators in $\mathcal{B}$. As before, Φ_A describes the local interactions on the finite set A i.e. it acts only on the degrees of freedom in the subset A. We can now define a class of norms for interactions.

Definition 3.7 (r-norm)**.** *Let* $r > 0$*. The* r*-norm for an interaction is defined as:*

$$\|\Phi\|_r = \sum_{X \ni 0} e^{r|X|} \|\Phi_X\|.$$

To avoid interactions which "explode" as the number of sites interacting or the distance between interacting sites grows, we focus only on interactions with $\|\Phi\|_r < \infty$ for some r. For this class of models the infinite volume dynamics exists, as discussed in the following theorem, see [41] Theorem III.3.6.

Theorem 3.1 (Existence of the infinite volume dynamics)**.** *If* $\|\Phi\|_r < \infty$ *for some* $r > 0$*, then for any* $t \in \mathbb{R}$ *there exists an automorphism* α_t *s.t.*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \|\alpha_t^\Lambda(a) - \alpha_t(a)\| = 0 \quad \forall a \in \mathcal{B}.$$

*The convergence is uniform in t on any bounded interval and the limit is taken over any sequence of van Hove volumes (e.g. concentric increasing boxes in $\mathbb{Z}^d$). The family α_t, $t \in \mathbb{R}$ constitutes a one-parameter group of *-automorphisms on $\mathcal{B}$.*

Infinite volume Gibbs states are positive normalised linear functionals which reproduce the property of Eq. (3.2) through the so called KMS condition.

Definition 3.8 (Infinite volume Gibbs states)**.** *Let α_t describe the infinite volume dynamics for the interaction Φ. An infinite volume Gibbs state with interaction Φ is a normalised, positive linear functional $\langle\cdot\rangle_\beta : \mathcal{B} \to \mathbb{R}$ such that for any $a, b \in \mathcal{B}$ there exists an analytic function $\mathfrak{F} : \{z \in \mathbb{C} : \, 0 \leq Im\, z \leq \beta\} \to \mathbb{R}$ such that*

$$\mathfrak{F}(t) = \langle a\, \alpha_t(b)\rangle_\beta, \;\; \mathfrak{F}(t + i\beta) = \langle \alpha_t(b)\, a\rangle_\beta.$$

States fulfilling this property take also the name of KMS states, from the names of Kubo [46], Martin and Schwinger [54].

As in the classical case, for the rest of the chapter we focus on translation invariant states. The set of KMS states at inverse temperature β for the interaction Φ is denoted by $\mathcal{K}_\beta(\Phi)$.

Proposition 3.1. *$\mathcal{K}_\beta(\Phi)$ is not empty and convex.*

Proof. Convexity is clear from Def. 3.8. Let us prove $\mathcal{K}_\beta(\Phi) \neq \emptyset$. For any $\Lambda \subset\subset \mathbb{Z}^d$ let $\langle\cdot\rangle_{\Lambda,\beta}$ be the finite volume Gibbs state for the interaction Φ. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a sequence of increasing volumes, i.e. $\Lambda_n \subset \Lambda_{n+1}$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} \Lambda_n = \mathbb{Z}^d$. Any cluster point for the sequence of finite volume states $\{\langle\cdot\rangle_{\Lambda_n,\beta}\}_{n\in\mathbb{N}}$ would naturally be a KMS state, since any finite volume state fulfils Eq. (3.2). We show that at least one such cluster point exists.

Notice that there exists $\tilde{\mathcal{B}} \subset \mathcal{B}$ which is dense in $\mathcal{B}$ and countable. $\tilde{\mathcal{B}}$ is then a countable collection of observables $\tilde{\mathcal{B}} = \{a_j\}_{j\in\mathbb{N}}$. We restrict our proof to these operators – the general case follows by continuity. We proceed now with the following diagonal argument.

- Consider the sequence $\{\langle a_1\rangle_{\Lambda_n,\beta}\}_{n\in\mathbb{N}}$. Since it is bounded we can extract a converging subsequence $\{\langle a_1\rangle_{\Lambda^1_{n_1},\beta}\}_{n_1\in\mathbb{N}}$ with $\{\Lambda^1_{n_1}\}_{n_1\in\mathbb{N}} \subset \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Lambda^1_{n_1} \nearrow \mathbb{Z}^d$.
- Consider the sequence $\{\langle a_2\rangle_{\Lambda^1_{n_1},\beta}\}_{n_1\in\mathbb{N}}$. It is a bounded sequence, thus we can extract a converging subsequence $\{\langle a_2\rangle_{\Lambda^2_{n_2},\beta}\}_{n_2\in\mathbb{N}}$ with $\{\Lambda^2_{n_2}\}_{n_2\in\mathbb{N}} \subset \{\Lambda^1_{n_1}\}_{n_1\in\mathbb{N}}$ $\subset \{\Lambda_n\}_{n\in\mathbb{N}}$ and $\Lambda^2_{n_2} \nearrow \mathbb{Z}^d$.

- This procedure can be iterated for all a_j, $j \in \mathbb{N}$. For each of them we can find a sequence of increasing volumes $\{\Lambda^j_{n_j}\}_{n_j\in\mathbb{N}}$ such that $\lim_{n_j\to\infty}\Lambda^j_{n_j} = \mathbb{Z}^d$ and $\{\Lambda^j_{n_j}\}_{n_j\in\mathbb{N}} \subset \{\Lambda^{(j-1)}_{n_{j-1}}\}_{n_{j-1}\in\mathbb{N}} \subset \cdots \subset \{\Lambda^1_{n_1}\}_{n_1\in\mathbb{N}} \subset \{\Lambda_n\}_{n\in\mathbb{N}}$, with $\{\langle a_j\rangle_{\Lambda_{n_j},\beta}\}_{n_j\in\mathbb{N}}$ converging.
- Define the sequence of increasing volume $\{\tilde{\Lambda}_m\}_{m\in\mathbb{N}}$ such that $\tilde{\Lambda}_1 = \Lambda^1_1$, $\tilde{\Lambda}_2 = \Lambda^2_2, \ldots,\ \tilde{\Lambda}_k = \Lambda^k_k$, i.e. the k-th element of $\{\tilde{\Lambda}_m\}_{m\in\mathbb{N}}$ is the k-th element of $\{\Lambda^k_{n_k}\}_{n_k\in\mathbb{N}}$. Then $\{\langle a_j\rangle_{\tilde{\Lambda}_m,\beta}\}_{m\in\mathbb{N}}$ converges as $m\to\infty$ for any $j\in\mathbb{N}$.

□

Definition 3.9. *If $|\mathcal{K}_\beta(\Phi)| > 1$, the system with interaction Φ undergoes a phase transition at inverse temperature β.*

Clearly $\mathcal{K}_\beta(\Phi)$ in the quantum framework plays a similar role to $\mathcal{G}_\beta(\Phi)$ in the classical one. Since $\mathcal{K}_\beta(\Phi)$ is a convex set, it is possible to define extremal states precisely as in the classical setting.

Definition 3.10 (Extremal Gibbs states)**.** *Let $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$. It is called extremal if any convex decomposition of the form $\langle\cdot\rangle_\beta = t\langle\cdot\rangle'_\beta + (1-t)\langle\cdot\rangle''_\beta$ with $t\in(0,1)$ and $\langle\cdot\rangle'_\beta$, $\langle\cdot\rangle''_\beta \in \mathcal{K}_\beta(\Phi)$ implies $\langle\cdot\rangle'_\beta = \langle\cdot\rangle''_\beta = \langle\cdot\rangle_\beta$.*

We denote by ex$\mathcal{K}_\beta(\Phi)$ the set of extremal KMS states for the interaction Φ at inverse temperature β. Extremal states decomposition is possible also in the quantum case, as stated in the following theorem.

Theorem 3.2 (Extremal states decomposition)**.** *For any $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$ there exists a unique measure ν over $\mathcal{K}_\beta(\Phi)$ concentrated on ex$\mathcal{K}_\beta(\Phi)$ such that $\langle\cdot\rangle_\beta$ is the barycentre of ν.*

For a proof of this result see [41], Theorem IV.3.3. We conclude by remarking that extremal states have some *cluster properties*, just as in the classical setting – see Theorem 2.2.

Theorem 3.3 (Cluster properties for extremal KMS states)**.** *Let $\langle\cdot\rangle_\beta \in$ex$\mathcal{K}_\beta(\Phi)$. Then for any $a, b\in\mathcal{B}$*

$$\lim_{\|x\|\to\infty} \langle a\,\theta_x b\rangle_\beta = \langle a\rangle_\beta\langle b\rangle_\beta,$$

with θ_x the translation by a vector $x\in\mathbb{Z}^d$.

3.3 Symmetries and Mermin-Wagner Theorem

In this section we discuss symmetries and Mermin-Wagner Theorem in the quantum setting. Most of the discussion is similar to the one of the classical case of Section 2.3.

Definition 3.11 (Symmetry). *Let $\{\Phi_X\}_{X\subset\subset\mathbb{Z}^d}$ be an interaction for a quantum lattice system with set of quasi-local observables $\mathcal{B}$. A symmetry for this model is a group G with a unitary representation $\{V_g\}_{g\in G}$, $V_g \in \mathcal{B}$ for each $g \in G$ such that*

$$V_g \Phi_X V_g^* = \Phi_X \quad \forall g \in G,\ \forall X \subset\subset \mathbb{Z}^d.$$

Example 3.4. Let $\{\Phi_X\}_{X\subset\subset\mathbb{Z}^d}$ be the interaction for the ferromagnetic Heisenberg model, i.e. the one described in Ex. 3.3 with $J_{xy} = J > 0$ for any $(x, y) \in \mathcal{E}$. This model is SU(2) invariant. For any $X \subset\subset \mathbb{Z}^d$, given $U_X = \prod_{x\in X} \mathrm{e}^{i\alpha_1 \mathcal{S}_x^1 + i\alpha_2 \mathcal{S}_x^2 + i\alpha_3 \mathcal{S}_x^3}$ we have

$$U_X \Phi_X U_X^* = \Phi_X \tag{3.3}$$

for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Example 3.5. Let $\{\Phi_X\}_{X\subset\subset\mathbb{Z}^d}$ be the interaction for the ferromagnetic XXZ model, i.e. the one described in Ex. 3.3 with $J_{xy}^1 = J_{xy}^2 \neq J_{xy}^3$ for any $(x, y) \in \mathcal{E}$. This model is U(1) invariant. Indeed for any finite subset X, given $U_X = \prod_{x\in X} \mathrm{e}^{i\alpha\mathcal{S}_x^3}$, we have

$$U_X \Phi_X U_X^* = \Phi_X \tag{3.4}$$

for any $X \subset\subset \mathbb{Z}^d$ and $\alpha \in \mathbb{R}$.

Given $g \in G$ and $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$, we denote $\langle\cdot\rangle_\beta^g$ the linear functional such that

$$\langle a\rangle_\beta^g = \langle V_g a V_g^*\rangle_\beta \tag{3.5}$$

for any $a \in \mathcal{B}$.

Proposition 3.2. *Let Φ be a G-symmetric interaction and let $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$. Then $\langle\cdot\rangle_\beta^g \in \mathcal{K}_\beta(\Phi)$ for any $g \in G$.*

Proof. By the definition of $\langle\cdot\rangle_\beta^g$, for any $a, b \in \mathcal{B}$

$$\langle ab\rangle_\beta^g = \langle V_g a V_g^* V_g b V_g^*\rangle_\beta. \tag{3.6}$$

By Def. 3.8 there exists $\mathfrak{F} : \{z \in \mathbb{C} : 0 \le \operatorname{Im} z \le \beta\} \to \mathbb{R}$ such that

$$\mathfrak{F}(t) = \langle V_g a V_g^* \alpha_t(V_g b V_g^*)\rangle_\beta, \quad \mathfrak{F}(t+i\beta) = \langle \alpha_t(V_g^* b V_g) V_g a V_g^*\rangle_\beta. \tag{3.7}$$

Notice that by Def.s 3.5, 3.11 and Theorem 3.1 for any $a \in \mathcal{B}$

$$\alpha_t(V_g a V_g^*) = V_g \alpha_t(a) V_g^*. \tag{3.8}$$

This implies that

$$\langle V_g a V_g^* \alpha_t(V_g b V_g^*)\rangle_\beta = \langle a\, \alpha_t(b)\rangle_\beta^g, \tag{3.9}$$

$$\langle \alpha_t(V_g^* b V_g) V_g a V_g^*\rangle_\beta = \langle \alpha_t(b)\, a\rangle_\beta^g. \tag{3.10}$$

The statement is thus proved. □

Definition 3.12 (Spontaneous symmetry breaking)**.** *Let Φ be a G-invariant interaction. If there exists $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$ such that*

$$\langle\cdot\rangle_\beta^g \neq \langle\cdot\rangle_\beta$$

the system undergoes a spontaneous symmetry breaking at inverse temperature β.

Remark. As for the classical case, notice that spontaneous symmetry breaking is a particular instance of phase transitions. Indeed, when $|\mathcal{K}_\beta(\Phi)| = 1$ and $\mathcal{K}_\beta(\Phi) = \{\langle\cdot\rangle_\beta\}$, then $\langle\cdot\rangle_\beta = \langle\cdot\rangle_\beta^g$ for any $g \in G$ since all the states coincide. In order to have spontaneous symmetry breaking, it is necessary that $|\mathcal{K}_\beta(\Phi)| > 1$, i.e. a phase transition must be taking place.

We mention now the celebrated Mermin Wagner-Theorem. We have seen it for the classical case in Chapter 2. Recall that this theorem states that there can not be spontaneous symmetry breaking of a continuous symmetry in 1d and 2d systems. The first work in this direction was originally formulated by Mermin and Wagner [56] for the quantum Heisenberg model, but we take the approach by Fröhlich an Pfister [28, 26]. We state it for two-body interactions, but it can be generalised to the many-body case.

Theorem 3.4. *Let $\{\Phi_X\}_{X\subset\subset\mathbb{Z}^2}$ be a two-body interaction, i.e. $\Phi_X = 0$ if $|X| \neq 2$. Assume $\sum_{X\ni 0}\|\Phi_X\| < \infty$. Let G be a compact connected Lie group constituting a symmetry for the model. If there exists $C > 0$ such that*

$$\|\Phi_{xy}\| \leq C\frac{1}{|x-y|^4} \;\; \forall\, x, y \in \mathbb{Z}^2$$

then for any $\beta \in (0, \infty)$ *and any* $\langle \cdot \rangle_\beta \in \mathcal{K}_\beta(\Phi)$

$$\langle \cdot \rangle_\beta = \langle \cdot \rangle_\beta^g \quad \forall g \in G.$$

See [28] for a proof of this statement.

Chapter 4

Decay of correlations in 2d quantum systems

This chapter is devoted to the absence of spontaneous symmetry breaking of continuous symmetry in $d = 2$, and the behaviour of correlation functions when this happens. Section 4.1 provides a brief discussion of the role of Mermin-Wagner Theorem in the literature and of the relationship between the absence of spontaneous symmetry breaking and the decay of the relevant correlation functions. The subsequent sections are adapted from [8] and are devoted to the description of the results present in that paper. Namely, we identify a wide class of models with U(1) symmetry which exhibit power law decay of correlations on any 2d graph.

4.1 Some remarks on Mermin-Wagner Theorem

In the previous Chapters we took the point of view of defining spontaneous symmetry breaking via some group-invariance properties (or lack thereof) of KMS states. In the literature, though, the expression *spontaneous symmetry breaking* has taken various mathematical definitions with the same physical flavour (i.e. the appearance of "order" at low temperature). The same is true for Mermin Wagner-Theorem. Indeed, in its original formulation [56], Mermin-Wagner Theorem looks quite different from the versions we mentioned in Chapters 2 and 3 – what was actually proved is that the quantum Heisenberg model at low dimensionality exhibits no spontaneous magnetisation.

In the following years many papers were produced with different results concerning the absence of spontaneous breaking of a continuous symmetry in $d = 1, 2$. From a physical point of view, many of these results are equivalent, in the sense that they describe the same physical phenomenon. On the other hand, this is not always the case from a mathematical point of view. The term *Mermin-Wagner Theorem* came thus to denote a great variety of different mathematical statements all pointing to the same notion that there is no spontaneous breaking of a continuous symmetry in low dimensional systems. In this chapter we are particularly concerned with the notion of decay of correlations.

From a practical point of view, studying spontaneous symmetry breaking of explicit quantum lattice models only via the theoretical definition of KMS states as outlined in Chapter 3 can be complicated and for many models there are not many rigorous results about $\mathcal{K}_\beta(\Phi)$, even when the physical intuition is clear. An alternative approach is to study some properly chosen two-point correlation functions which mirror the symmetry of the model – the absence of spontaneous symmetry breaking is signalled by their decay as the distance between the two sites involved increases. In order to justify this rigorously in the next paragraph we briefly discuss a straightforward result – namely, the absence of spontaneous symmetry breaking as defined in Def. 3.12 implies the decay of the relevant correlations (see for example [49]).

4.1.1 Absence of spontaneous symmetry breaking implies decay of correlations

In this paragraph we restrict ourselves to translation invariant models – an assumption we drop in the subsequent sections. We follow an approach similar to [49], Theorem 10.7. Let us consider an interaction $\Phi = \{\Phi_A\}_{A \subset\subset \mathbb{Z}^d}$ with a symmetry group G. Let $\mathcal{K}_\beta(\Phi)$ be the set of its translation invariant Gibbs states. We define an order parameter for the symmetry group G as follows.

Definition 4.1 (Order parameter)**.** *An order parameter for the symmetry group G is an observable $M \in \mathcal{B}$ such that for any $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$*

$$\langle M\rangle_\beta = 0$$

if and only if $\langle\cdot\rangle_\beta = \langle\cdot\rangle_\beta^g$ for any $g \in G$.

A similar definition of order parameter holds also in the classical setting, with $M : \Omega \to \mathbb{R}$ some local function instead of a linear operator.

Remark. To any $x \in \Lambda$ we can associate an order parameter $M_x = \theta_x M$, where θ_x is the translation by a vector x. Due to translation invariance $\langle M_x\rangle_\beta = \langle M\rangle_\beta$ for any $\langle\cdot\rangle_\beta \in \mathcal{K}_\beta(\Phi)$.

From a physical point of view, the role of the order parameter is to signal the presence of spontaneous symmetry breaking. It is clear from the definition that it should be some operator which is not invariant under the action of the group G.

Example 4.1. Recall the Ising model defined in section 2.3. This model is $\mathbb{Z}_2$ invariant. A natural order parameter is σ_x for some $x \in \mathbb{Z}^d$. Its expectation value takes the name of *magnetisation*.

Example 4.2. Recall the XXZ model introduced in Ex. 3.3. We have seen in Ex. 3.5 that it is U(1) symmetric. The natural order parameters for the models would be $\mathcal{S}_x^1$ and $\mathcal{S}_x^2$ for any $x \in \mathbb{Z}^d$. The intuition is simple: the U(1) invariance of the XXZ model is given by spin "rotations" along the third axis.

Theorem 4.1. *Assume there is no spontaneous symmetry breaking, i.e.*

$$\langle \cdot \rangle_\beta = \langle \cdot \rangle_\beta^g \quad \forall g \in G, \, \forall \langle \cdot \rangle_\beta \in \mathcal{K}_\beta(\Phi).$$

Then, for any $\langle \cdot \rangle_\beta \in \mathcal{K}_\beta(\Phi)$

$$\lim_{d(x,y)\to\infty} \langle M_x M_y \rangle_\beta = 0.$$

Proof. The proof is a straightforward application of the cluster properties of extremal states described in Theorem 3.3. Let $\langle \cdot \rangle_\beta \in \mathcal{K}_\beta(\Phi)$. Let us label the elements of $\mathrm{ex}\mathcal{K}_\beta(\Phi)$ with elements of some set $A_{\Phi,\beta}$ endowed with a suitable σ-algebra $\mathscr{B}_{\Phi,\beta}$ (i.e. there is a bijection between $\mathrm{ex}\mathcal{K}_\beta(\Phi)$ and $A_{\Phi,\beta}$ so that $\langle \cdot \rangle_\beta^\alpha \in \mathrm{ex}\mathcal{K}_\beta(\Phi)$ for any $\alpha \in A_{\Phi,\beta}$, as described in the Remark after Theorem 2.1 in the classical case). By the extremal states decomposition (Theorem 3.2) there exists a measure ν over $(A_{\Phi,\beta}, \mathscr{B}_{\Phi,\beta})$ such that for any x, $y \in \Lambda$

$$\langle M_x M_y \rangle_\beta = \int_{A_{\Phi,\beta}} d\nu(\alpha) \langle M_x M_y \rangle_\beta^\alpha. \tag{4.1}$$

By the cluster properties of extremal states:

$$\begin{aligned}
\lim_{d(x,y)\to\infty} \langle M_x M_y \rangle_\beta &= \lim_{d(x,y)\to\infty} \int_{A_{\Phi,\beta}} d\nu(\alpha) \langle M_x M_y \rangle_\beta^\alpha \\
&= \int_{A_{\Phi,\beta}} d\nu(\alpha) \lim_{d(x,y)\to\infty} \langle M_x M_y \rangle_\beta^\alpha \\
&= \int_{A_{\Phi,\beta}} d\nu(\alpha) \left(\langle M \rangle_\beta^\alpha \right)^2 \\
&= 0.
\end{aligned} \tag{4.2}$$

The last line follows from the definition of order parameter M. The result is thus proved. $\square$

Notice that an analogous statement clearly holds for classical states as well for a similarly defined order parameter. Indeed the proof of the theorem above is based on features of KMS states that are common to classical Gibbs states as well – namely, extremal states decomposition and cluster properties of extremal states.

4.2 Decay of correlations in 2d lattice systems

As we have seen, the absence of spontaneous symmetry breaking is related to the decay of the relevant two point correlation function of the model both from a physical and a mathematical point of view. In our paper [8] we focus on the rate of this decay in 2d models with continuous symmetry. This topic has been of interest for a long time, and many results concerning specific models are present in the literature.

Fisher and Jasnow [21] provided a logarithmic upper bound for the decay of the relevant correlations of the quantum Heisenberg model. McBryan and Spencer [55] proved power-law decay of correlation functions for the classical rotor model in a short and lucid article that exploits a method known as *complex rotations*. Shlosman obtained similar results with a different method [73]. Power-law decay was established for some quantum systems in [13, 42]; these proofs use the Fourier transform and the Bogoliubov inequality, and they are limited to regular two-dimensional lattices. A more general result was obtained by Koma and Tasaki using complex rotations [45]; the latter proof was simplified and applied to the XXZ spin-$\frac{1}{2}$ model on generic two-dimensional lattices in in [30]. The articles [13, 21, 30, 42, 45] are all formulated for specific models and they rely on explicit settings. But the method of proof of [45] is robust and it is clear to experts that it should apply much more broadly.

In our work [8] we identify a very general class of quantum models which enjoy a U(1) symmetry and exhibit a power-law bound for the decay of correlations. As a consequence, we get new results for generalised Heisenberg models with higher spins; for the Hubbard model; for the t-J model; and for random loop models. Our results are detailed in the next sections. In particular, Section 4.3 is devoted to the statement of our result for the specific models just mentioned. In Section 4.4 we state and prove our general theorem, which exploits the complex rotations method. In Section 4.5 we provide the proofs of the statements for the specific models mentioned in Section 4.3.

4.3 Results for specific models

This section is devoted to the statement of our results from [8] for some specific models. In particular, we investigate SU(2)-invariant quantum spins (Section 4.3.1), random loop models (Section 4.3.2), the Hubbard model (Section 4.3.3), and the t-J model (Section 4.3.4).

4.3.1 Quantum spin systems

Let Λ be a finite graph with set of edges $\mathcal{E}$. We consider graphs of arbitrary sizes, but with bounded *perimeter constant* γ:

$$\gamma = \max_{x\in\Lambda} \max_{\ell\in\mathbb{N}} \frac{1}{\ell} \big| \big\{ y \in \Lambda \,|\, d(x,y) = \ell \big\} \big|. \tag{4.3}$$

Typical examples of allowed graphs are finite subsets of $\mathbb{Z}^2$ where edges are between nearest-neighbours, in which case $\gamma = 4$, or finite subsets of the triangular, hexagonal, or kagomé lattices. It is worth pointing out that we do not assume translation invariance.

Let $s \in \frac{1}{2}\mathbb{N}$, and let $\vec{\mathcal{S}} = \big(\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^3\big)$ be the vector of spin-s matrices acting on the Hilbert Space $\mathbb{C}^{2s+1}$. Moreover we define the ladder operators $\mathcal{S}^{\pm} = \mathcal{S}^1 \pm i\mathcal{S}^2$.

The most general SU(2) invariant hamiltonian with spin s and pair interactions is of the form

$$H_\Lambda = - \sum_{(x,y)\in\mathcal{E}} \sum_{k=1}^{2s} c_k(x,y) \left(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y \right)^k. \tag{4.4}$$

Here, $c_k(x,y)$ denotes the coupling constants, and $\mathcal{S}^i_x = \mathcal{S}^i \otimes \mathbb{1}_{\Lambda\setminus x}$. The hamiltonian acts on the Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^{2s+1}$. The corresponding Gibbs state at inverse temperature β is $\langle\cdot\rangle_{\Lambda,\beta} = \mathrm{Tr}\ \cdot \mathrm{e}^{-\beta H_\Lambda} / \mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda}$. This is a generalisation of the Heisenberg model introduced in Ex. 3.3.

We assume without loss of generality that

$$\sum_k |c_k(x,y)| \left(3s^2\right)^k \leq 1. \tag{4.5}$$

for all $x, y \in \Lambda$. Being SU(2) invariant, the hamiltonian of this model commutes with the total spin along any of the three axes. We actually have

$$\left[\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y,\, \mathcal{S}^i_x + \mathcal{S}^i_y \right] = 0 \tag{4.6}$$

for $i = 1, 2, 3$. As a matter of fact, we only exploit one of these symmetries to get an algebraic bound for the decay of correlations.

Theorem 4.2. *Let H_Λ be the hamiltonian defined in* (4.4)*. There exist $C > 0$ and $\xi(\beta) > 0$ (the latter depending on β, γ, s, but not on $x, y \in \Lambda$) such that*

$$|\langle \mathcal{S}_x^j \mathcal{S}_y^j \rangle_{\Lambda,\beta}| \leq C \, (d(x, y) + 1)^{-\xi(\beta)} \, .$$

More generally, if $\mathcal{O}_y \in \mathcal{B}_y$, we have

$$|\langle \mathcal{S}_x^+ \mathcal{O}_y \rangle_{\Lambda,\beta}| \leq C \, (d(x, y) + 1)^{-\xi(\beta)} \, .$$

Further, we have

$$\lim_{\beta \to \infty} \beta \, \xi(\beta) = (32 s \gamma^2)^{-1}.$$

We could also consider models with interactions that are asymmetric with respect to spin directions; if the model retains a U(1) symmetry, the theorem and its proof can be readily adapted [30]. In this case we get the correct behaviour for $\xi(\beta)$. Indeed, a Berezinski-Kosterlitz-Thouless transition should take place where the decay of correlations changes from exponential to power law, with exponent behaving as β^{-1} for large β. This was proved in the classical XY model [29]. For models with SU(2) symmetry, one expects exponential decay for all positive temperatures.

The proof of Theorem 4.2 can be found in Section 4.5.

4.3.2 Random loop models

Models of random loops have been introduced as representations of quantum spin systems [77, 4, 81] and they are increasingly popular in probability theory. A special case is the *random interchange model* where the outcomes are permutations given by products of random transpositions. We obtain an explicit theorem about decay of loop correlations that looks natural in the context of quantum spins, but are rather surprising in the probabilistic context.

To each edge of the graph $(\Lambda, \mathcal{E})$ is attached the "time" interval $[0, \beta]$. Independent Poisson point processes result in the occurrences of "crosses" with intensity u and "double bars" with intensity $1 - u$, where $u \in [0, 1]$ is a parameter. This means that, on the edge $(x, y) \in \mathcal{E}$ and in the infinitesimal time interval $[t, t + \mathrm{d}t] \subset [0, \beta]$, a cross appears with probability $u\mathrm{d}t$, a double bar appears with probability $(1 - u)\mathrm{d}t$, and there is nothing with probability $1 - \mathrm{d}t$. We denote by ρ the measure and by ω its realisations.

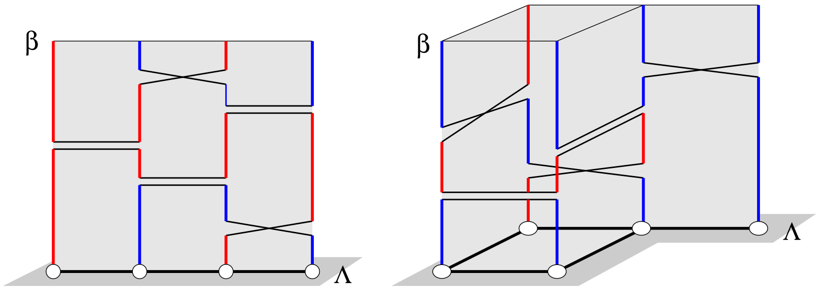

Figure 4.1: Illustrations for the random loop models. The vertices all lie in the horizontal plane and random crosses and double bars occur in the "time" intervals $[0, \beta]$ on top of each edges. In both of these examples, the realisations have exactly two loops, denoted in red and blue. This figure is from [81, 8].

Given a realisation ω, loops are formed by the close trajectories obtained by travelling along the time direction, with periodic conditions, and with jumps on the neighbour whenever a cross or a double bar is present. If it is a cross, the trajectory continues in the same direction; if it is a double bar, the trajectory continues in the opposite direction. See the illustration in Fig. 4.1. We let $\mathcal{L}(\omega)$ denote the set of loops of the realisation ω; notice that $|\mathcal{L}(\omega)| < \infty$ with probability 1.

Let $\theta > 0$ another parameter. The partition function of the model is given by

$$Z_{\Lambda,\beta}^{\theta,u} = \int \theta^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega). \tag{4.7}$$

The relevant measure is

$$\mu_{\Lambda,\beta}^{\theta,u}(\mathrm{d}\omega) = \frac{1}{Z_{\Lambda,\beta}^{\theta,u}} \theta^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega). \tag{4.8}$$

The special case $u = 1$ and $\theta = 1$ is the random interchange model; crosses give transpositions, and the loops are equivalent to permutation cycles.

We obtain the following result about the probability $\mathbb{P}_{\Lambda,\beta}^{\theta,u}(x \leftrightarrow y)$ of two sites x, y belonging to the same loop.

Theorem 4.3. *Let $\theta = 2, 3, 4, \ldots,$ $u \in [0, 1]$ and $x, y \in \Lambda$. There exist $C > 0$ and $\xi(\beta) > 0$ (the latter depending on β, γ, θ, u but not on $x, y \in \Lambda$) such that*

$$\mathbb{P}_{\Lambda,\beta}^{\theta,u}(x \leftrightarrow y) \leq C \left(d(x, y) + 1\right)^{-\xi(\beta)},$$

Further, we have

$$\lim_{\beta\to\infty} \beta\, \xi(\beta) = \left[8\gamma^2(\theta-1)^2(u+(1-u)\theta+1)\right]^{-1}.$$

Since this model is closely related to a class of quantum spin systems, the theorem follows from our general theorem on quantum systems with continuous symmetry, Theorem 4.6. See Section 4.5 for the details.

4.3.3 The Hubbard Model

Let $(\Lambda, \mathcal{E})$ be a graph, and let us define the usual fermionic creation and annihilation operators for spin-$\frac{1}{2}$ on each site $x \in \Lambda$, $c^\dagger_{\sigma,x}$, $c_{\sigma,x}$, $\sigma = \uparrow, \downarrow$. The Hilbert space for one site is defined as $\mathcal{H}_x = \mathrm{span}\{0, \uparrow, \downarrow, \uparrow\downarrow\} \simeq \mathbb{C}^4$. The creation and annihilation operators satisfy the usual anticommutation relations, $\left\{c_{\sigma,x}, c_{\sigma',y}\right\} = \delta_{xy}\delta_{\sigma\sigma'}$.

The Hubbard model is a model of hopping electrons. Here, we consider a general case with possibly long-range hoppings. The hamiltonian is

$$H_\Lambda = -\frac{1}{2} \sum_{x,y\in\Lambda} \sum_{\sigma=\uparrow,\downarrow} t_{xy} \left(c^\dagger_{\sigma,x} c_{\sigma,y} + c^\dagger_{\sigma,y} c_{\sigma,x}\right) + V\left(\{n_{\sigma,x}\}\right). \tag{4.9}$$

Here, the number operators are defined in the usual way: $n_{\uparrow,x} = c^\dagger_{\uparrow,x} c_{\uparrow,x}$, $n_{\downarrow,x} = c^\dagger_{\downarrow,x} c_{\downarrow,x}$, $n_x = n_{\uparrow,x} + n_{\downarrow,x}$. $V\left(\{n_{\sigma,x}\}\right)$ is a generic potential depending only on the total number of particles of any possible spin per site. For background on the Hubbard model, we recommend the excellent review by Lieb [52].

This hamiltonian enjoys an SU(2) symmetry with generators

$$\mathcal{S}^+_x = c^\dagger_{\uparrow,x} c_{\downarrow,x}, \qquad \mathcal{S}^-_x = (\mathcal{S}^+_x)^\dagger, \qquad \mathcal{S}^3_x = \tfrac{1}{2}(n_{\uparrow,x} - n_{\downarrow,x}). \tag{4.10}$$

We are interested mostly in the symmetry U(1)$\subset$SU(2) given by the conservation of spin along the third axis:

$$\Big[\sum_{\sigma=\uparrow,\downarrow} t_{xy} \left(c^\dagger_{\sigma,x} c_{\sigma,y} + c^\dagger_{\sigma,y} c_{\sigma,x}\right), \mathcal{S}^3_x + \mathcal{S}^3_y\Big] = 0. \tag{4.11}$$

Moreover, the general Hubbard hamiltonian (4.9) conserves the number of particles. This provides another U(1) symmetry, namely,

$$\Big[\sum_{\sigma=\uparrow,\downarrow} t_{xy} \left(c^\dagger_{\sigma,x} c_{\sigma,y} + c^\dagger_{\sigma,y} c_{\sigma,x}\right), \sum_{\sigma=\uparrow,\downarrow} (n_{\sigma,x} + n_{\sigma,y})\Big] = 0. \tag{4.12}$$

Different symmetries lead to the decay of different correlation functions. We focus on three two-point functions:

(i) $\langle c^\dagger_{\uparrow,x} c_{\downarrow,x} c^\dagger_{\downarrow,y} c_{\uparrow,y}\rangle_{\Lambda,\beta}$ that represents magnetic long-range order;

(ii) $\langle c^\dagger_{\uparrow,x} c^\dagger_{\downarrow,x} c_{\uparrow,y} c_{\downarrow,y}\rangle_{\Lambda,\beta}$ that is related to Cooper pairs and superconductivity;

(iii) $\langle c^\dagger_{\sigma,x} c_{\sigma,y}\rangle_{\Lambda,\beta}$, the off-diagonal long-range order.

The last two quantities have been studied in [45, 53]. In [45] the decay is studied with a method similar to ours, under the condition that $t_{xy} = 0$ if $d(x,y) \geq R$ for some positive R. In [53] it is assumed that t_{xy} decays fast enough, i.e. $t_{xy} \sim t\, d(x,y)^{-\alpha}$ with $\alpha > 4$ and t some constant. We show that we need to work in this same setting for the general result in Theorem 4.6 to be applicable.

Theorem 4.4. *Let H_Λ be the hamiltonian of the Hubbard model* (4.9) *defined on Λ, and $x, y \in \Lambda$. Suppose that $t_{xy} = t(d(x,y)+1)^{-\alpha}$ with $\alpha > 4$. Then there exist $C > 0$, $\xi(\beta) > 0$ (the latter depending on β, γ, α, t, but not on $x, y \in \Lambda$) such that*

$$\left.\begin{array}{c} |\langle c^\dagger_{\uparrow,x} c_{\downarrow,x} c^\dagger_{\downarrow,y} c_{\uparrow,y}\rangle_{\Lambda,\beta}| \\ |\langle c^\dagger_{\uparrow,x} c^\dagger_{\downarrow,x} c_{\uparrow,y} c_{\downarrow,y}\rangle_{\Lambda,\beta}| \\ |\langle c^\dagger_{\sigma,x} c_{\sigma,y}\rangle_{\Lambda,\beta}| \end{array}\right\} \leq C(d(x,y)+1)^{-\xi(\beta)}$$

where $\sigma \in \{\uparrow, \downarrow\}$ in the last line. Further, we have

$$\lim_{\beta\to\infty} \beta\, \xi(\beta) = \Big(64\gamma^2 |t| \sum_{r\geq 1} r^{-\alpha+3}\Big)^{-1}.$$

Notice that the theorem above provides an algebraic bound for the decay of correlations for any $\alpha > 4$. As explained in the proof (Section 4.5), this requirement is necessary in order to ensure the finiteness of the K-norm of the interaction (see Eq. (4.22)) independently from the size of Λ.

4.3.4 The t-J model

A well known variant of the Hubbard model is given by the t-J model, which is described by the following hamiltonian.

$$H_\Lambda = -\frac{t}{2} \sum_{(x,y)\in\mathcal{E}} \sum_{\sigma=\uparrow,\downarrow} \Big(c^\dagger_{\sigma,x} c_{\sigma,y} + c^\dagger_{\sigma,y} c_{\sigma,x}\Big) + J \sum_{(x,y)\in\mathcal{E}} \Big(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y - \frac{1}{4} n_x n_y\Big). \quad (4.13)$$

In the hamiltonian above, the parameters t, J are real numbers, and

$$\mathcal{S}_x^i = \frac{1}{2} \sum_{\sigma,\mu=\uparrow,\downarrow} c_{\sigma,x}^\dagger \tau_{\sigma,\mu}^i c_{\mu,x}, \tag{4.14}$$

with $i = \{1,2,3\}$ and τ^1, τ^2, τ^3 the three Pauli matrices describing spin $\frac{1}{2}$ (see Ex. 3.1). Explicitly,

$$\begin{aligned} \mathcal{S}_x^1 =& \tfrac{1}{2}\big(c_{\uparrow,x}^\dagger c_{\downarrow x} + c_{\downarrow,x}^\dagger c_{\uparrow y}\big), \\ \mathcal{S}_x^2 =& -\tfrac{\mathrm{i}}{2}\big(c_{\uparrow,x}^\dagger c_{\downarrow x} - c_{\downarrow,x}^\dagger c_{\uparrow y}\big), \\ \mathcal{S}_x^3 =& \tfrac{1}{2}(n_{\uparrow,x} - n_{\downarrow,x}). \end{aligned} \tag{4.15}$$

These are precisely the generators of SU(2) defined previously for the Hubbard model, see Eq. (4.10). This model conserves the number of particles and the spin along the third axis — i.e. it enjoys U(1) symmetries: For all $x, y \in \Lambda$,

$$\begin{aligned} &\Big[-\tfrac{t}{2} \sum_{\sigma=\uparrow,\downarrow} \Big(c_{\sigma,x}^\dagger c_{\sigma,y} + c_{\sigma,y}^\dagger c_{\sigma,x}\Big) + J\big(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y - \tfrac{1}{4} n_x n_y\big), \sum_{\sigma=\uparrow,\downarrow} n_{\sigma,x} + n_{\sigma,y}\Big] = 0, \\ &\Big[-\tfrac{t}{2} \sum_{\sigma=\uparrow,\downarrow} \Big(c_{\sigma,x}^\dagger c_{\sigma,y} + c_{\sigma,y}^\dagger c_{\sigma,x}\Big) + J\big(\vec{\mathcal{S}}_x \cdot \vec{\mathcal{S}}_y - \tfrac{1}{4} n_x n_y\big), \frac{1}{2}(n_{\uparrow,x} - n_{\downarrow,x} + n_{\uparrow,y} - n_{\downarrow,y})\Big] = 0. \end{aligned} \tag{4.16}$$

The analysis proposed for the Hubbard Model holds for the t-J model as well, and we estimate the decay of several correlation functions.

Theorem 4.5. *Let H_Λ be the hamiltonian of the t-J model* (4.13) *defined on Λ and let $x, y \in \Lambda$. Then there exist $C > 0$ and $\xi(\beta) > 0$ (the latter depending on β, γ, t, J, but not on $x, y \in \Lambda$) such that*

$$\left.\begin{array}{c} |\langle c_{\uparrow,x}^\dagger c_{\downarrow,x} c_{\downarrow,y}^\dagger c_{\uparrow,y}\rangle_{\Lambda,\beta}| \\ |\langle c_{\uparrow,x}^\dagger c_{\downarrow,x}^\dagger c_{\uparrow,y} c_{\downarrow,y}\rangle_{\Lambda,\beta}| \\ |\langle c_{\sigma,x}^\dagger c_{\sigma,y}\rangle_{\Lambda,\beta}| \end{array}\right\} \leq C\,(d(x,y)+1)^{-\xi(\beta)}$$

with $\sigma \in \{\uparrow,\downarrow\}$ in the last line. Further,

$$\lim_{\beta\to\infty} \beta\,\xi(\beta) = (128\gamma^2\,(2|t| + |J|))^{-1}.$$

4.4 General model with U(1) symmetry

Let $(\Lambda, \mathcal{E})$ denote a finite graph, with Λ the set of vertices and $\mathcal{E}$ the set of edges, and perimeter constant γ, see Eq. (4.3). Let $\mathcal{H}_\Lambda$ be a finite-dimensional Hilbert space (we have in mind the tensor product $\otimes_{x\in\Lambda}\mathbb{C}^N$, but we do not need to assume this explicitly). Let $\mathcal{B}(\mathcal{H}_\Lambda)$ denote the algebra of linear operators on $\mathcal{H}_\Lambda$. We assume the existence of sub-algebras $\mathcal{B}_A \subset \mathcal{B}(\mathcal{H}_\Lambda)$, with the properties that $\mathbb{1} \in \mathcal{B}_A$ for all $A \subset \Lambda$ and $\mathcal{B}_A \subseteq \mathcal{B}_{A'}$ whenever $A \subseteq A' \subseteq \Lambda$, and of hermitian operators $\{S_x\}_{x\in\Lambda}$ that obey the following commutation relations:

(a) For any $x, y \in \Lambda$ with $x \neq y$

$$[S_x, S_y] = 0. \tag{4.17}$$

(b) For any $x \in \Lambda$, $A \subset \Lambda$, and $\Psi \in \mathcal{B}_A$

$$[S_x, \Psi] \begin{cases} = 0 & \text{if } x \notin A; \\ \in \mathcal{B}_A & \text{if } x \in A. \end{cases} \tag{4.18}$$

The hamiltonian is the sum of "local" terms. Precisely, we assume that

$$H_\Lambda = \sum_{A\subset\Lambda} \Phi_A, \tag{4.19}$$

where the operators Φ_A are hermitian and they belong to $\mathcal{B}_A$. The hamiltonian satisfies a U(1) symmetry with generator $\sum_x S_x$ in the sense that

$$\Big[\Phi_A, \sum_{x\in A} S_x\Big] = 0 \tag{4.20}$$

for all $A \subset \Lambda$. Without loss of generality, we assume that for all $x \in \Lambda$,

$$\|S_x\| = 1. \tag{4.21}$$

We introduce a norm for the interactions that depends on a parameter $K \geq 0$, namely,

$$\|\Phi\|_K = \sup_{y\in\Lambda} \sum_{\substack{A\subset\Lambda \\ \text{s.t. } y\in A}} \|\Phi_A\|(|A|-1)^2(\operatorname{diam}(A)+1)^{2K(|A|-1)+2}. \tag{4.22}$$

Notice that this K-norm does not take into account possible one-body terms.

As usual, the Gibbs state $\langle\cdot\rangle_{\Lambda,\beta}$ is the linear functional that assigns the value

$$\langle a\rangle_{\Lambda,\beta} = \frac{\mathrm{Tr}\ a\mathrm{e}^{-\beta H_\Lambda}}{\mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda}} \tag{4.23}$$

to each operator $a \in \mathcal{B}(\mathcal{H}_\Lambda)$.

Furthermore, we assume the existence of a *correlation function* $O_{xy} \in \mathcal{B}_{\{x,y\}}$ for some $x, y \in \Lambda$, that satisfies the following relation; there exists $c \in \mathbb{R}$ such that

$$[S_x, O_{xy}] = cO_{xy}. \tag{4.24}$$

Notice that there are no assumptions about the commutator between S_y and O_{xy}. We are now ready to state a general version of the McBryan-Spencer-Koma-Tasaki Theorem [55, 45], which establishes power-law decay of two-point correlation functions for this wide class of models.

Theorem 4.6. *Suppose that* $\{S_x\}_{x\in\Lambda}$*,* $\{\Phi_A\}_{A\subset\Lambda}$*, and* O_{xy} *satisfy the properties* (4.17)–(4.21) *and* (4.24)*. Then there exist* $C > 0$ *and* $\xi(\beta) > 0$ *(uniform with respect to* Λ *and* $x, y \in \Lambda$*) such that*

$$|\langle O_{xy}\rangle_{\Lambda,\beta}| \le C\,(d(x,y)+1)^{-\xi(\beta)}\,.$$

Moreover, if there exists $K > 0$ *such that* $\|\Phi\|_K$ *is bounded uniformly in* Λ*, then*

$$\lim_{\beta\to\infty} \beta\,\xi(\beta) = \frac{c^2}{8\gamma\|\Phi\|_0}.$$

The rest of this section is devoted to the proof of this theorem. We follow the method of Koma and Tasaki that was used in the context of the Hubbard model [45]. Notice that the bound provided is independent of Λ – if the infinite volume limit of $\langle\cdot\rangle_{\Lambda,\beta}$ is well defined and exists, the bound holds also for the limiting infinite volume Gibbs state.

Proof. The proof is based on so-called complex rotations. Other necessary ingredients are the Trotter formula (see Appendix A) and a generalisation of Hölder inequality for matrices (see Appendix B), as in [30]. Let us define a real number for each site of the lattice, $\{\theta_z\}_{z\in\Lambda} \in \mathbb{R}^\Lambda$, namely,

$$\theta_z = \begin{cases} \kappa \log \frac{d(x,y)+1}{d(x,z)+1} & \text{if } d(x,z) \le d(x,y), \\ 0 & \text{otherwise,} \end{cases} \tag{4.25}$$

where κ is an arbitrary positive parameter. The complex rotation operator is

$$R = \prod_{y \in \Lambda} \mathrm{e}^{\theta_y S_y}. \tag{4.26}$$

For each subset $A \subseteq \Lambda$, we let $x_0(A)$ be the site (or one of the sites) in A that is at minimal distance from x. Using (4.20), we have

$$\begin{aligned} R^{-1} H_\Lambda R &= \sum_{A \subset \Lambda} \mathrm{e}^{-\sum_{y \in A}(\theta_y - \theta_{x_0(A)})S_y} \Phi_A \mathrm{e}^{\sum_{y \in A}(\theta_y - \theta_{x_0(A)})S_y} \\ &= \mathrm{e}^{-T_A} \Phi_A \mathrm{e}^{T_A}, \end{aligned} \tag{4.27}$$

where

$$T_A = \sum_{y \in A} (\theta_y - \theta_{x_0}) S_y. \tag{4.28}$$

Recall the notation $\mathrm{ad}_a(b) = [a, b]$. We use the multicommutator expansion to get

$$\begin{aligned} R^{-1} H_\Lambda R &= \sum_{A \subset \Lambda} \Phi_A + \sum_{j \geq 1} \sum_{A \subset \Lambda} \frac{(-1)^j}{j!} \mathrm{ad}^j_{T_A}(\Phi_A) \\ &= H_\Lambda + B + C, \end{aligned} \tag{4.29}$$

where

$$B = -\sum_{j \geq 1} \sum_{A \subset \Lambda} \frac{1}{(2j-1)!} \mathrm{ad}^{2j-1}_{T_A}(\Phi_A) \tag{4.30}$$

and

$$C = \sum_{j \geq 1} \sum_{A \subset \Lambda} \frac{1}{(2j)!} \mathrm{ad}^{2j}_{T_A}(\Phi_A). \tag{4.31}$$

B contains the terms of the multicommutator expansion odd in T_A and is thus anti-hermitian; C contains the terms even in T_A and is thus hermitian. Eq.s (4.24) and (4.25) imply that

$$R^{-1} O_{xy} R = \mathrm{e}^{-c\theta_x} O_{xy}. \tag{4.32}$$

We now apply the Trotter formula (see Appendix A, Theorem A.1) and Hölder inequality for matrices (see Appendix B, Definition B.1 to recall the explicit form of $\|\cdot\|_p$ with $p \geq 1$ and Theorem B.2 for the inequality itself). We get

$$
\begin{aligned}
\left|\operatorname{Tr} O_{xy} \mathrm{e}^{-\beta H_\Lambda}\right| &= \left|\operatorname{Tr} R^{-1} O_{xy} R \mathrm{e}^{-\beta R^{-1} H_\Lambda R}\right| \\
&= \mathrm{e}^{-c\theta_x} \left|\operatorname{Tr} O_{xy} \mathrm{e}^{-\beta H_\Lambda - \beta B - \beta C}\right| \\
&\leq \mathrm{e}^{-c\theta_x} \lim_{n\to\infty} \left|\operatorname{Tr} O_{xy} \left(\mathrm{e}^{-\frac{\beta}{n} H_\Lambda} \mathrm{e}^{-\frac{\beta}{n} B} \mathrm{e}^{-\frac{\beta}{n} C}\right)^n\right| \\
&\leq \mathrm{e}^{-c\theta_x} \lim_{n\to\infty} \|O_{xy}\|_\infty \| \mathrm{e}^{-\frac{\beta}{n} H_\Lambda}\|_n^n \, \|\mathrm{e}^{-\frac{\beta}{n} B}\|_\infty^n \, \|\mathrm{e}^{-\frac{\beta}{n} C}\|_\infty^n.
\end{aligned}
\tag{4.33}
$$

Notice that $\|\mathrm{e}^{-\frac{\beta}{n} B}\|_\infty = 1$ because B is anti-hermitian. From here onwards we use the standard notation $\|\cdot\|$ to denote the sup-norm $\|\cdot\|_\infty$. Since $\|\mathrm{e}^{-\frac{\beta}{n} H_\Lambda}\|_n^n = \operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}$, by Eq. (4.25) we obtain

$$
|\langle O_{xy}\rangle_{\Lambda,\beta}| \leq \mathrm{e}^{-c\kappa \log(d(x,y)+1)} \|O_{xy}\| \mathrm{e}^{\beta\|C\|}. \tag{4.34}
$$

We have to estimate $\|C\|$. Using $\|[A,B]\| \leq 2\|A\|\,\|B\|$, we have

$$
\begin{aligned}
\|C\| &\leq \sum_{j\geq 1} \sum_{A\subset\Lambda} \frac{2^{2j}}{(2j)!} \|T_A\|^{2j} \|\Phi_A\| \\
&= \sum_{A\subset\Lambda} \|\Phi_A\| \big(\cosh\left(2\|T_A\|\right) - 1\big).
\end{aligned}
\tag{4.35}
$$

We now use the inequality $\cosh u - 1 \leq \frac{1}{2} u^2 \mathrm{e}^u$, that is easily verified for all $u \geq 0$. We find

$$
\begin{aligned}
\|C\| &\leq 2 \sum_{A\subset\Lambda} \|\Phi_A\| \|T_A\|^2 \mathrm{e}^{2\|T_A\|} \\
&\leq 2 \sum_{A\subset\Lambda} \|\Phi_A\| \left(|A|-1\right) \mathrm{e}^{2\sum_{z\in A\setminus\{x_0(A)\}} |\theta_z - \theta_{x_0(A)}|} \sum_{z\in A\setminus\{x_0(A)\}} |\theta_z - \theta_{x_0(A)}|^2.
\end{aligned}
\tag{4.36}
$$

In the equation above we estimated $\|T_A\|^2$ using Cauchy-Schwarz, namely

$$
\|T_A\|^2 \leq \left(\sum_{y\in A\setminus\{x_0\}} |\theta_y - \theta_{x_0}|\right)^2 \leq \left(|A|-1\right) \sum_{y\in A} |\theta_y - \theta_{x_0}|^2. \tag{4.37}
$$

Then, by the explicit form of the $\{\theta_z\}_{z\in\Lambda}$ in Eq. (4.25),

$$\begin{aligned}\|C\| &\leq 2 \sum_{\substack{A\subset\Lambda \text{ s.t.}\\ d(x,x_0(A))\leq d(x,y)}} \|\Phi_A\|(|A|-1)(\operatorname{diam}(A)+1)^{2\kappa(|A|-1)} \sum_{z\in A\setminus\{x_0(A)\}} |\theta_z-\theta_{x_0(A)}|^2 \\ &\leq 2\kappa^2 \sum_{\substack{A\subset\Lambda \text{ s.t.}\\ d(x,x_0(A))\leq d(x,y)}} \|\Phi_A\|(|A|-1)^2(\operatorname{diam}(A)+1)^{2\kappa(|A|-1)+2}\frac{1}{(d(x,x_0(A))+1)^2}.\end{aligned} \tag{4.38}$$

We can now estimate $\|C\|$ further by reorganising the sums and using the definition of $\|\Phi\|_\kappa$ from Eq. (4.22).

$$\begin{aligned}\|C\| &\leq 2\kappa^2 \sum_{\substack{x_0\in\Lambda \text{ s.t.}\\ d(x,x_0)\leq d(x,y)}} \frac{1}{(d(x,x_0)+1)^2}\sum_{A\ni x_0}\|\Phi_A\|(|A|-1)^2(\operatorname{diam}(A)+1)^{2\kappa(|A|-1)+2} \\ &\leq 2\kappa^2 \sum_{\substack{x_0\in\Lambda \text{ s.t.}\\ d(x,x_0)\leq d(x,y)}} \frac{1}{(d(x,x_0)+1)^2}\|\Phi\|_\kappa.\end{aligned} \tag{4.39}$$

Recall the definition of the perimeter constant γ in Eq. (4.3). Since we consider only graphs $(\Lambda,\mathcal{E})$ such that it is finite we have

$$\begin{aligned}\|C\| &\leq 2\kappa^2\|\Phi\|_\kappa\left(\sum_{r=1}^{d(x,y)}\frac{\gamma r}{(1+r)^2}+1\right) \\ &\leq 2\kappa^2\|\Phi\|_\kappa\left(\sum_{r=1}^{d(x,y)}\frac{\gamma}{r}+1\right) \\ &\leq 2\kappa^2\gamma\|\Phi\|_\kappa\log(d(x,y)+1)+2\kappa^2\|\Phi\|_\kappa.\end{aligned} \tag{4.40}$$

We conclude that for all $\kappa>0$

$$|\langle O_{xy}\rangle_{\Lambda,\beta}| \leq \mathcal{C}_\kappa(d(x,y)+1)^{-(\kappa c-2\kappa^2\gamma\|\Phi\|_\kappa\beta)}. \tag{4.41}$$

with $\mathcal{C}_\kappa = \|O_{xy}\|\mathrm{e}^{2\beta\kappa^2\|\Phi\|_\kappa}$ and c the constant defined in Eq. (4.24). We would like to check that the power is $\sim\frac{1}{\beta}$ for β large enough. Choosing $\kappa=\frac{K}{\beta}$, the exponent in the above equation is

$$\xi_K(\beta) = \tfrac{K}{\beta}\big(c-2K\gamma\|\Phi\|_{\frac{K}{\beta}}\big). \tag{4.42}$$

Recall that in the last part of Theorem 4.6 it is assumed that there is a constant $\tilde{K}$ such that the $\tilde{K}$-norm of the interaction Φ converges, independently of Λ. Applying dominated convergence to $\|\Phi\|_K$ we then get

$$\lim_{\beta\to\infty} \beta\,\xi_K(\beta) = Kc - 2K^2\gamma\|\Phi\|_0. \tag{4.43}$$

The optimal value of K is $K^* = c/(4\gamma\|\Phi\|_0)$. We define $\xi(\beta) = \xi_{K^*}(\beta)$ and substitute $\mathcal{C}_\kappa$ with $\mathcal{C}_{\frac{K^*}{\beta}}$ in Eq.(4.41). This completes the proof. $\square$

4.5 Applications of the general theorem to the explicit examples

This section is devoted to the proofs of the various theorems stated in Sections 4.3.1–4.3.3 for some models of interest. They are all straightforward applications of Theorem 4.6.

Proof of Theorem 4.2. The interaction defining the hamiltonian has finite K-norm for any $K > 0$:

$$\|\Phi\|_K = 2^{2K+2} \sup_{z\in\Lambda} \sum_{\substack{w\,\text{s.t.}\\ d(w,z)=1}} \left\| \sum_{l=1}^{2s} c_l(w,z)(\vec{\mathcal{S}}_w \cdot \vec{\mathcal{S}}_z)^l \right\| \leq 2^{2K+2}\gamma. \tag{4.44}$$

The bound follows from the triangular inequality and the assumption in Eq. (4.5). Let $S_x = \frac{1}{s}\mathcal{S}_x^3$. It is bounded with norm 1 and $S_x + S_y$ commutes with the local hamiltonian so it provides the U(1) symmetry of Eq. (4.6). Let $O_{xy} = \mathcal{S}_x^+\mathcal{O}_y$ for some $\mathcal{O}_y \in \mathcal{B}_y$. It is bounded and

$$[S_x, O_{xy}] = s^{-1}O_{xy}. \tag{4.45}$$

Then, the value of c as defined in Theorem 4.6, Eq. (4.24), is $c = s^{-1}$. The result is now a straightforward application of Theorem 4.6. Consider $\xi_K(\beta)$ as defined in the proof of the general Theorem 4.6, Eq. (4.42). We get from Eq. (4.44)

$$\xi_K(\beta) \geq \tfrac{K}{\beta}\big(s^{-1} - 8K\gamma^2 2^{\frac{2K}{\beta}}\big) = \tilde{\xi}_K(\beta). \tag{4.46}$$

It is clear that $\lim_{\beta\to\infty} \beta\,\tilde{\xi}_K(\beta) = \frac{K}{s} - 8K^2\gamma^2$. By optimising with respect to K, we get the first statement of the theorem by defining $\xi(\beta) = \tilde{\xi}_{K^*}(\beta)$ where K^* is the optimal value of K. Due to the SU(2) invariance of the model, Eq. (4.6), we have

$$\langle \mathcal{S}_x^1 \mathcal{S}_y^1 \rangle_{\Lambda,\beta} = \langle \mathcal{S}_x^2 \mathcal{S}_y^2 \rangle_{\Lambda,\beta} = \langle \mathcal{S}_x^3 \mathcal{S}_y^3 \rangle_{\Lambda,\beta}. \tag{4.47}$$

From the definition of $\mathcal{S}_x^+$ and $\mathcal{S}_x^-$ it is easy to verify that

$$\langle \mathcal{S}_x^+ \mathcal{S}_y^- \rangle_{\Lambda,\beta} = 2\langle \mathcal{S}_x^1 \mathcal{S}_y^1 \rangle_{\Lambda,\beta} = 2\langle \mathcal{S}_x^2 \mathcal{S}_y^2 . \rangle_{\Lambda,\beta} \tag{4.48}$$

The first statement is then just a special case of the second one. □

Proof of Theorem 4.3. For θ an integer larger than 1, the loop model is equivalent to a quantum spin model [77, 4, 81]. Let $\theta = 2s + 1$ with $s \in \frac{1}{2}\mathbb{N}$. We introduce operators acting on $\mathbb{C}^\theta \otimes \mathbb{C}^\theta$, namely,

$$\begin{aligned} &T\, e_i \otimes e_j = e_i \otimes e_j, \\ &(e_i \otimes e_j, Q\, e_l \otimes e_k) = \delta_{i,j}\delta_{l,k}, \end{aligned} \tag{4.49}$$

where $\{e_j\}_{j=1}^{\theta}$ denotes the canonical basis of $\mathbb{C}^\theta$. Then we consider the Hilbert space $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^\theta$, and the hamiltonian

$$H_\Lambda = - \sum_{(x,y)\in\mathcal{E}} \left(uT_{xy} + (1-u)Q_{xy} - 1 \right). \tag{4.50}$$

Here, T_{xy} actually means $T \otimes \mathbb{1}_{\Lambda\setminus\{x,y\}}$, and analogously for Q_{xy}. The corresponding Gibbs state at inverse temperature β is $\langle\cdot\rangle_{\Lambda,\beta} = \mathrm{Tr}\ \cdot \mathrm{e}^{-\beta H_\Lambda}/\mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda}$. It can be shown that for any $u \in [0,1]$, the partition function $Z_{\Lambda,\beta}^{\theta,u}$ in Eq. (4.7) is equal to the quantum partition function, namely

$$Z_{\Lambda,\beta}^{\theta,u} = \mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda}. \tag{4.51}$$

It can be shown ([81], Lemma 3.1) that for any $x, y \in \Lambda$,

$$\begin{aligned} &[T_{xy}, \mathcal{S}_x^i + \mathcal{S}_y^i] = 0, \quad i = 1,2,3 \\ &[Q_{xy}, \mathcal{S}_x^2 + \mathcal{S}_y^2] = 0. \end{aligned} \tag{4.52}$$

This implies that we can define $S_x = \frac{1}{s}\mathcal{S}_x^2$ and the hamiltonian has the right commutation relations with it. Moreover it is easy to check that the interaction has finite norm for any $K > 0$:

$$\|\Phi\|_K = 2^{2K+2} \sup_{y\in\Lambda} \sum_{\substack{x \,\text{s.t.}\\ d(x,y)=1}} \|uT_{xy} + (1-u)Q_{xy} - 1\| \leq 2^{2K+2}\gamma\left(u + (1-u)\theta + 1\right). \tag{4.53}$$

The bound follows from the triangular inequality and from $\|T\| = 1$, $\|Q\| = \theta$. Let $\mathcal{Q}^{\pm} = \mathcal{S}^1 \pm i\mathcal{S}^3$. Then for any $x, y \in \Lambda$ and $\mathcal{O} \in \mathcal{B}_y$,

$$\left[S_x, \mathcal{Q}_x^{+}\mathcal{O}_y\right] = s^{-1}\mathcal{Q}_x^{+}\mathcal{O}_y, \tag{4.54}$$

i.e. $c = s^{-1}$ in Theorem 4.6.

The general statement of Theorem 4.6 can be applied to the correlation function $\langle \mathcal{Q}_x^{+}\mathcal{Q}_y^{-}\rangle_{\Lambda,\beta}$. Indeed, let $\xi_K(\beta)$ be as defined in the proof of Theorem 4.6, Eq. (4.42). From Eq. (4.53), we have

$$\xi_K(\beta) \geq \frac{K}{\beta}\left(\frac{2}{\theta - 1} - 8K\gamma^2 2^{\frac{2K}{\beta}}(u + (1-u)\theta + 1)\right) = \tilde{\xi}_K(\beta). \tag{4.55}$$

Moreover, we have

$$\lim_{\beta\to\infty} \beta\,\tilde{\xi}_K(\beta) = \frac{2K}{\theta - 1} - 8K^2\gamma^2\left(u + \theta(1-u) + 1\right). \tag{4.56}$$

Optimising with respect to K and definining $\xi(\beta) = \tilde{\xi}_{K^*}(\beta)$ where K^* is the optimal value of K, one finds the result for $\langle \mathcal{Q}_x^{+}\mathcal{Q}_y^{-}\rangle_{\Lambda,\beta}$.

Due to the symmetry of the model,

$$\langle \mathcal{S}_x^1\mathcal{S}_y^1\rangle_{\Lambda,\beta} = \langle \mathcal{S}_x^3\mathcal{S}_y^3\rangle_{\Lambda,\beta}. \tag{4.57}$$

Then, by the definition of $\mathcal{Q}^{\pm}$,

$$\langle \mathcal{Q}_x^{+}\mathcal{Q}_y^{-}\rangle_{\Lambda,\beta} = 2\langle \mathcal{S}_x^1\mathcal{S}_y^1\rangle_{\Lambda,\beta} = 2\langle \mathcal{S}_x^3\mathcal{S}_y^3\rangle_{\Lambda,\beta}. \tag{4.58}$$

Thus the result is proved for the correlations $\langle \mathcal{S}_x^3\mathcal{S}_y^3\rangle_{\Lambda,\beta}$. The theorem regarding the probability of two sites being connected follows from

$$\langle \mathcal{S}_x^3\mathcal{S}_y^3\rangle_{\Lambda,\beta} = \frac{1}{12}(\theta^2 - 1)\mathbb{P}_{\Lambda,\beta}^{\theta,u}(x \leftrightarrow y). \tag{4.59}$$

See [81], Theorem 3.3, for a proof of this statement. □

Proof of Theorem 4.4. It can be easily checked that the K-norm of the interaction associated to the hamiltonian is

$$\|\Phi\|_K = \sup_{z\in\Lambda} \sum_{w\in\Lambda} \left\| \sum_\sigma c^\dagger_{\sigma,z} c_{\sigma,w} + c^\dagger_{\sigma,w} c_{\sigma,z} \right\| |t| (d(z,w)+1)^{-(\alpha-2K-2)}. \tag{4.60}$$

First, notice that the one body potential $V(\{n_{\sigma,x}\})$ does not play any role. Secondly, we would like the norm to be independent of Λ, i.e. finite no matter what the size of Λ is. By the triangular inequality and given the definition of γ,

$$\|\Phi\|_K \leq 2|t|\gamma \sum_{r\geq 1} r^{-(\alpha-2K-3)}. \tag{4.61}$$

Notice that for any $\alpha > 4$ there exists $K > 0$ such that $\alpha > 2K + 4$. This ensures the existence of values of K such that $\|\Phi\|_K$ converges, as required by Theorem 4.6.

Let us focus on the first two-point function. Let $S_x = (n_{\uparrow,x} - n_{\downarrow,x})$, that is $2\mathcal{S}^3_x$ according to Eq. (4.10). It is a bounded operator which commutes with the local hamiltonian; since the Hubbard Model is SU(2) invariant, it enjoys the U(1) symmetry given by the conservation of the total 'spin' along the third axis, see Eq. (4.11). Let $O_{xy} = c^\dagger_{\uparrow,x} c_{\downarrow,x} \mathcal{O}_y$ with $\mathcal{O}_y \in \mathcal{B}_y$

$$[S_x, O_{xy}] = 2O_{xy}, \tag{4.62}$$

so we can take $c = 2$ in Theorem (4.6).

Let us now focus on the second correlation function. As seen in Section 4.3.3, the hamiltonian enjoys a U(1) symmetry due to the conservation of the number of particles, see Eq. (4.12), so we have $S_x = \frac{1}{2}(n_{\uparrow,x} + n_{\downarrow,x})$.

Now, let $O_{xy} = c^\dagger_{\uparrow,x} c^\dagger_{\downarrow,x} \mathcal{O}_y$ for some $\mathcal{O}_y \in \mathcal{B}_y$. Then

$$[S_x, O_{xy}] = O_{xy}, \tag{4.63}$$

and $c = 1$ for Theorem 4.6.

For the third case we also use $S_x = \frac{1}{2}(n_{\uparrow,x} + n_{\downarrow,x})$. Let $O_{xy} = c^\dagger_{\sigma,x} \mathcal{O}_y$ for any possible value of σ and any $\mathcal{O}_y \in \mathcal{B}_y$. Then

$$[S_x, O_{xy}] = \frac{1}{2} O_{xy}. \tag{4.64}$$

The theorem is now a straightforward application of Theorem 4.6. Indeed, let $\xi_K(\beta)$ be defined as in Eq. (4.42). Then it is clear that, in all three cases,

$$\xi_K(\beta) \geq \frac{K}{2\beta}\left(1 - 8K\gamma^2|t|\tau^{\frac{2K}{\beta}}\right) = \tilde{\xi}_K(\beta) \tag{4.65}$$

where $\tau^{\frac{2K}{\beta}} = \sum_{r\geq 1} r^{-\alpha+3+\frac{2K}{\beta}}$. Then by dominated convergence,

$$\lim_{\beta\to\infty} \beta\,\tilde{\xi}_K(\beta) = \frac{K}{2}(1 - 8K\gamma^2|t|\tau^0). \tag{4.66}$$

Optimising with respect to K and defining $\xi(\beta) = \tilde{\xi}_{K^*}(\beta)$ where K^* is the optimal value of K, one finds the result (choosing $\mathcal{O}_y$ equal to $c^\dagger_{\downarrow,y}c_{\uparrow,y}$ in the first case, to $c_{\uparrow,y}c_{\downarrow,y}$ in the second case, and to $c_{\sigma,y}$ in the third case).

Notice that $\alpha > 4$ is needed for τ^0 to be well defined. □

Proof of Theorem 4.5. The interaction defining the t-J model has finite K-norm for any value of K and it can be explicitly evaluated:

$$\|\Phi\|_K = 2^{2K+2}\sup_{x\in\Lambda}\sum_{x\sim y}\Big\| -\tfrac{t}{2}\sum_{\sigma}(c^\dagger_{\sigma,x}c_{\sigma,y} + c^\dagger_{\sigma,y}c_{\sigma,x}) + J\big(\vec{\mathcal{S}}_x\cdot\vec{\mathcal{S}}_y - \tfrac{1}{4}n_x n_y\big)\Big\|. \tag{4.67}$$

We can bound $\|\Phi\|_K$ using the triangular inequality; by the definition of γ,

$$\|\Phi\|_K \leq 2^{2K+2}\gamma\,(2|t| + |J|)\,. \tag{4.68}$$

For the first correlation function, let $S_x = (n_{\uparrow,x} - n_{\downarrow,x})$. It commutes with the local hamiltonian, see the second equation in (4.16), and is bounded with norm equal to 1. Let $O_{xy} = c^\dagger_{\uparrow,x}c_{\downarrow,x}\mathcal{O}_y$ with $\mathcal{O}_y \in \mathcal{B}_y$. It fulfills

$$[S_x, O_{xy}] = 2O_{xy}. \tag{4.69}$$

Then $c = 2$ in Theorem 4.6.

For the second correlation function, let $S_x = n_{\uparrow,x} + n_{\downarrow,x}$. The hamiltonian conserves the number of particles, see the U(1) symmetry in the first line of Eq. (4.16), so this operator commutes with the local hamiltonian. Let $O_{xy} = c^\dagger_{\uparrow,x}c^\dagger_{\downarrow,x}\mathcal{O}_y$ with $\mathcal{O}_y \in \mathcal{B}_y$; then

$$[S_x, O_{xy}] = O_{xy}. \tag{4.70}$$

The value of the constant c in Theorem 4.6 is then 1.

For the third case, let $S_x = \frac{1}{2}(n_{\uparrow,x} + n_{\downarrow,x})$ as before. Let $O_{xy} = c^\dagger_{\sigma,x}\mathcal{O}_y$ for any possible value of σ and $\mathcal{O}_y \in \mathcal{B}_y$; it fulfills

$$[S_x, O_{xy}] = \frac{1}{2}O_{xy}. \tag{4.71}$$

Then $c = \frac{1}{2}$ in Theorem 4.6.

The result is now a straightforward application of Theorem 4.6 to the three cases. Indeed, let $\xi_K(\beta)$ as defined in Eq. (4.42). Given the bound in Eq. (4.68) and the possible values of c, in all the three cases we have

$$\xi_K(\beta) \geq \frac{K}{2\beta}\left(1 - 16K\gamma^2(2|t| + |J|)2^{\frac{2K}{\beta}}\right) = \tilde{\xi}_K(\beta). \tag{4.72}$$

It is clear that

$$\lim_{\beta\to\infty} \beta\,\tilde{\xi}_K(\beta) = \frac{K}{2}(1 - 16K\gamma^2(2|t| + |J|)). \tag{4.73}$$

We get the claim by optimising with respect to K and by defining $\xi(\beta) = \tilde{\xi}_{K^*}(\beta)$, where K^* is the optimal value of K. We take $\mathcal{O}_y = c^\dagger_{\downarrow,y}c_{\uparrow,y}$ in the first case, $\mathcal{O}_y = c_{\uparrow,y}c_{\downarrow,y}$ in the second case, and $\mathcal{O}_y = c_{\sigma,y}$ in the last case. $\square$

Chapter 5

Correlation inequalities for classical and quantum XY models

Correlations inequalities have been of invaluable help in the study of classical models. They are an important tool in the investigation of infinite volume states and the behaviour of magnetisation of classical models. In this chapter we are particularly interested in the so called Griffiths-Ginibre inequalities. They were firstly discussed by Griffiths in his work about the Ising model [34], while a more general formulation was provided in the seminal work by Ginibre [32]. This last approach applies to a wide class of classical models of interest. While classical correlation inequalities have been object of intense research efforts, far less is known about their quantum counterparts, and for many relevant models in the literature there are no results on this subject.

The model we focus on is the XY model, both in the classical and quantum setting. The first section of this chapter is devoted to an introduction to Ginibre inequalities for the classical XY model, following our review paper [10]. In the second section we discuss the same inequalities for the quantum case. For the spin-$\frac{1}{2}$ case they were proved for a two-body interaction in [31] and then independently in a more general setting in [76, 67, 9]. In this last work we also prove them for the ground state of spin-1 models. This section is adapted from this paper and from the review [10].

The last section is devoted to possible applications of Griffiths inequalities in the quantum case. As mentioned in our review [10] a comparison between the critical temperatures of the quantum XY and Ising models can be provided [76, 67].

Moreover we prove some statements about infinite volume correlation functions, following [9]. We also provide some new results about quantum XY models with random couplings.

5.1 Correlation inequalities for the classical XY model: a review

The classical XY model is a particular instance of O(n) model ($n = 2$) introduced in Example 2.4 in its simplest setting. In this section, we look at a generalised XY model with many-body interactions. Let Λ be the set of sites hosting the spins. The configuration space of the system is defined as $\Omega_\Lambda = \{\{\sigma_x\}_{x\in\Lambda} : \sigma_x \in \mathbb{S}^1 \ \forall x \in \Lambda\}$: each site hosts a vector with unit length lying on a unit circle. It is convenient to represent the spins by means of angles, namely

$$\begin{aligned} \sigma_x^1 &= \cos\phi_x && (5.1)\\ \sigma_x^2 &= \sin\phi_x && (5.2) \end{aligned}$$

with $\phi_x \in [0, 2\pi)$. The energy of a configuration $\sigma \in \Omega_\Lambda$ with angles $\phi = \{\phi_x\}_{x\in\Lambda}$ is

$$H_\Lambda^{\mathrm{cl}}(\phi) = -\sum_{A\subset\Lambda} J_A^1 \prod_{x\in A} \sigma_x^1 + J_A^2 \prod_{x\in A} \sigma_x^2, \tag{5.3}$$

with $J_A^i \in \mathbb{R}$ for all $A \subset \Lambda$. The expectation value at inverse temperature β of a functional f on the configuration space is the usual one:

$$\langle f\rangle_{\Lambda,\beta}^{\mathrm{cl}} = \frac{1}{Z_{\Lambda,\beta}^{\mathrm{cl}}} \int d\phi\, \mathrm{e}^{-\beta H_\Lambda^{\mathrm{cl}}(\phi)}\, f(\phi), \tag{5.4}$$

where $Z_{\Lambda,\beta}^{\mathrm{cl}} = \int d\phi\, \mathrm{e}^{-\beta H_\Lambda^{\mathrm{cl}}(\phi)}$ is the partition function and $\int d\phi = \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{x\in\Lambda} \frac{d\phi_x}{2\pi}$. The following correlation inequalities hold for this model.

Theorem 5.1. *Assume that $J_A^1, J_A^2 \geq 0$ for all $A \subset \Lambda$. The following inequalities hold true for all $X, Y \subset \Lambda$, and for all $\beta > 0$.*

$$\begin{aligned} &\Big\langle \prod_{x\in X} \sigma_x^1 \prod_{x\in Y} \sigma_x^1 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} - \Big\langle \prod_{x\in X} \sigma_x^1 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} \Big\langle \prod_{x\in Y} \sigma_x^1 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} \geq 0,\\ &\Big\langle \prod_{x\in X} \sigma_x^1 \prod_{x\in Y} \sigma_x^2 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} - \Big\langle \prod_{x\in X} \sigma_x^1 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} \Big\langle \prod_{x\in Y} \sigma_x^2 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} \leq 0. \end{aligned}$$

We will see in Section 5.2 that equivalent results hold in the quantum case (Theorems 5.4, 5.5). The proof is given in Section 5.1.1. These inequalities are known as Ginibre inequalities — first introduced by Griffiths for the Ising model [34] and systematised by Ginibre [32] who provided a general framework for inequalities of this form. Ginibre inequalities for the classical XY model have then been established with different techniques [32, 48, 59, 47, 60]. A straightforward corollary of this theorem is monotonicity with respect to coupling constants, as we see now.

Corollary 5.1. *Assume that $J_A^1, J_A^2 \geq 0$ for all $A \subset \Lambda$. Then for all $X, Y \subset \Lambda$, and for all $\beta > 0$*

$$\frac{\partial}{\partial J_Y^1} \left\langle \prod_{x \in X} \sigma_x^1 \right\rangle_{\Lambda,\beta}^{\mathrm{cl}} \geq 0,$$
$$\frac{\partial}{\partial J_Y^2} \left\langle \prod_{x \in X} \sigma_x^1 \right\rangle_{\Lambda,\beta}^{\mathrm{cl}} \leq 0.$$

Monotonicity of correlations with respect to temperature does not follow straightforwardly from the corollary. This can nonetheless be proved for the classical XY model.

Theorem 5.2.

Assume that $J_A^1 \geq |J_A^2|$ for all $A \subset \Lambda$, and that $J_A^2 = 0$ whenever $|A|$ is odd. Then for all $A, B \subset \Lambda$, we have

$$\frac{\partial}{\partial \beta} \Big\langle \prod_{x \in B} \sigma_x^1 \Big\rangle_{\Lambda,\beta}^{\mathrm{cl}} \geq 0.$$

Let us restrict to the two-body case and assume that H_Λ^{cl} is given by

$$H_\Lambda^{\mathrm{cl}} = - \sum_{x,y \in \Lambda} J_{xy} \left(\sigma_x^1 \sigma_y^1 + \eta_{xy} \sigma_x^2 \sigma_y^2 \right). \tag{5.5}$$

Then if $|\eta_{xy}| \leq 1$ for all x, y,

$$\frac{\partial}{\partial J_{xy}} \left\langle \prod_{z \in A} \sigma_z^1 \right\rangle_{\Lambda,\beta}^{\mathrm{cl}} \geq 0.$$

This result has been proposed and discussed in various works [32, 60, 57] — see Section 5.1.1 for the details.

We now compare the correlations of the Ising and XY models. Recall from Ex. 2.3 that the configuration space of the Ising model is $\Omega_\Lambda^{\mathrm{Is}} = \{-1,1\}^\Lambda$, that is, Ising configurations are given by $\{\omega_x\}_{x\in\Lambda}$ with $\omega_x = \pm 1$ for each $x \in \Lambda$. We consider many-body interactions, so the energy of a configuration $\omega \in \Omega_\Lambda^{\mathrm{Is}}$ is

$$H^{\mathrm{Is}}_{\Lambda,\{J_A\}}(\omega) = -\sum_{A\subset\Lambda} J_A \prod_{x\in A} \omega_x; \tag{5.6}$$

we assume that the system is *ferromagnetic*, i.e. the coupling constants $J_A \geq 0$ are nonnegative. The Gibbs state at inverse temperature β is

$$\langle f\rangle^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta} = \frac{1}{Z^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta}} \sum_{\omega\in\Omega_\Lambda^{\mathrm{Is}}} f(\omega) \mathrm{e}^{-\beta H^{\mathrm{Is}}_{\Lambda,\{J_A\}}(\omega)}, \tag{5.7}$$

with f any functional on $\Omega_\Lambda^{\mathrm{Is}}$ and $Z^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta} = \sum_{\omega\in\Omega_\Lambda^{\mathrm{Is}}} \mathrm{e}^{-\beta H^{\mathrm{Is}}_{\Lambda,\{J_A\}}(\omega)}$ is the partition function. The following theorem holds [47].

Theorem 5.3. *Assume that $J_A^1, J_A^2 \geq 0$ for all $A \subset \Lambda$. Then for all $X \subset \Lambda$ and all $\beta > 0$,*

$$\left\langle \prod_{x\in X} \sigma_x^1 \right\rangle^{\mathrm{cl}}_{\Lambda,\beta} \leq \left\langle \prod_{x\in X} \omega_x \right\rangle^{\mathrm{Is}}_{\Lambda,\{J_A^1\},\beta}.$$

A review of the proofs is proposed in Section 5.1.1.

5.1.1 Proofs for the classical XY model

The proofs require several steps and additional lemmas. The following paragraphs are devoted to a complete study of their proofs.

Notation. Given local variables $\{\sigma_x\}_{x\in\Lambda}$ (e.g. Ising spins or components of XY spins), we define $\sigma_A = \prod_{x\in A} \sigma_x$ for $A \subset \Lambda$.

Griffiths inequalities for the Ising model, FKG inequalities, and proof of Theorem 5.1

We start with Theorem 5.1. We describe the approach proposed in [48, 47], and use a similar notation. Their framework relies on some well known properties of the Ising model and on the so called *FKG inequality*.

Lemma 5.1 (Griffiths inequalities for the Ising model)**.** *Let f and g be functionals on $\Omega_\Lambda^{\mathrm{Is}}$ such that they can be expressed as power series of $\prod_{x\in A} \omega_x$, $A \subset \Lambda$ with* *positive coefficients. Then*

$$\langle f\rangle^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta} \geq 0;$$
$$\langle fg\rangle^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta} \geq \langle f\rangle^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta}\langle g\rangle^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta}.$$

We do not provide the proof of this result — see [34, 32] for the original formulation and [23] for a modern description. An immediate consequence is the following.

Corollary 5.2. *Given f with the properties in Lemma 5.1, we have for any $A \subset \Lambda$*

$$\frac{\partial}{\partial J_A}\langle f\rangle^{\mathrm{Is}}_{\Lambda,\{J_A\},\beta} \geq 0.$$

Another result which is very useful in this framework is the so called FKG inequality. We formulate it in a specific setting. Let $\mathcal{I}_N = \left[0, \frac{\pi}{2}\right]^N$ for some $N \in \mathbb{N}$. Any $\psi \in \mathcal{I}_N$ is then a collection of angles $\psi = (\psi_1, \ldots, \psi_N)$. It is possible to introduce a partial ordering relation on $\mathcal{I}_N$ as follows: for any $\psi, \xi \in \mathcal{I}_N$, $\psi \leq \xi$ if and only if $\psi_i \leq \xi_i$ for all $i \in \{1, \ldots, N\}$. A function f on $\mathcal{I}_N$ is said to be increasing (or decreasing) if $\psi \leq \xi$ implies $f(\psi) \leq f(\xi)$ (or $f(\psi) \geq f(\xi)$) for all $\psi, \xi \in \mathcal{I}_N$. The following result holds.

Lemma 5.2 (FKG inequality)**.** *Let $d\nu(\psi) = p(\psi)\prod_{i=1}^N d\mu(\psi_i)$ be a normalised measure on $\mathcal{I}_N$, with $d\mu(\psi_i)$ a normalised measure on $\left[0, \frac{\pi}{2}\right]$, $p(\psi) \geq 0$ for all $\psi \in \mathcal{I}_N$ and*

$$p(\psi \vee \xi)p(\psi \wedge \xi) \geq p(\psi)p(\xi), \tag{5.8}$$

where $(\psi \vee \xi)_i = \max(\psi_i, \xi_i)$ and $(\psi \wedge \xi)_i = \min(\psi_i, \xi_i)$. Then for any f and g increasing (or decreasing) functions on $\mathcal{I}_N$

$$\int fg\, d\nu \geq \int f\, d\nu \int g\, d\nu.$$

The inequality changes sign if one of the functions is increasing and the other is decreasing.

We also skip the proof of this statement. We refer to [22] for the original result, to [69, 47] for the formulation above, and [23] for its relevance in the study of the Ising model.

Before turning to the actual proof of the theorem, we introduce another useful lemma.

Lemma 5.3. *Let* $\{q_x\}_{x\in\Lambda}$ *be a collection of positive increasing (decreasing) functions on* $\left[0, \frac{\pi}{2}\right]$*. Then for any* $\theta, \psi \in \mathcal{I}_{|\Lambda|}$ *and any* $A \subset \Lambda$*,*

$$q_A(\theta \vee \psi) + q_A(\theta \wedge \psi) \geq q_A(\psi) + q_A(\theta).$$

We do not provide the proof here, see [69, 47] for more details. We can now discuss the proof of Theorem 5.1.

Proof of Theorem 5.1. Since the temperature does not play any role in this section, we set $\beta = 1$ in the following and we drop any dependency on it. The main idea of the proof is to describe a classical XY spin as a pair of Ising spins and an angular variable. The new notation for $\sigma_x \in \mathbb{S}^1$ is

$$\begin{aligned} \sigma_x^1 &= \cos(\theta_x)U_x, &(5.9)\\ \sigma_x^2 &= \sin(\theta_x)V_x, &(5.10) \end{aligned}$$

with $U_x, V_x \in \{-1, 1\}$ for all $x \in \Lambda$ and $\theta = (\theta_{x_1}, \ldots, \theta_{x_\Lambda}) \in \mathcal{I}_{|\Lambda|}$. With this notation, it is possible to express H_Λ^{cl} of Eq. (5.3) as the sum of two Ising hamiltonians with spins $\{U_x\}_{x\in\Lambda}$, $\{V_x\}_{x\in\Lambda}$ respectively:

$$\begin{aligned} H_\Lambda^{\mathrm{cl}}(\theta, U, V) &= -\sum_{A\subset\Lambda}\left(J_A^1 \prod_{x\in A}\cos(\theta_x)U_A + J_A^2 \prod_{x\in A}\sin(\theta_x)V_A\right) &(5.11)\\ &= H^{\mathrm{Is}}_{\Lambda,\{\cos(\theta)_A J_A^1\}}(U) + H^{\mathrm{Is}}_{\Lambda,\{\sin(\theta)_A J_A^2\}}(V). &(5.12) \end{aligned}$$

Let us introduce the notation: $J_A^1 \prod_{x\in A}\cos(\theta_x) = \mathscr{J}_A(\theta)$, $J_A^2 \prod_{x\in A}\sin(\theta_x) = \mathscr{K}_A(\theta)$, $\int d\theta = \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \prod_{x\in\Lambda} \frac{2}{\pi} d\theta_x$. Then

$$\begin{aligned} \langle \sigma_X^1 \sigma_Y^1 \rangle_\Lambda^{\mathrm{cl}} &= \frac{\int d\theta\, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}} \cos(\theta)_X \cos(\theta)_Y \langle U_X U_Y \rangle^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}}}{\int d\theta\, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}}} \\ &\geq \frac{\int d\theta\, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}} \cos(\theta)_X \langle U_X \rangle^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} \cos(\theta)_Y \langle U_Y \rangle^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}}}{\int d\theta\, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}. \end{aligned}$$

The inequality above follows from Lemma 5.1. Moreover

$$\langle \sigma_X^1 \sigma_Y^2 \rangle_\Lambda^{\rm cl} = \frac{\int d\theta Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}} \cos(\theta)_X \langle U_X \rangle^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}} \sin(\theta)_Y \langle V_Y \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}{\int d\theta\, Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}.$$

$\cos(\theta)_X$ and $\sin(\theta)_X$ are respectively decreasing and increasing on $\mathcal{I}_{|\Lambda|}$ for any $X \subset \Lambda$. Let us now consider $\langle U_X \rangle^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}}$. By Corollary 5.2, it is a decreasing function on $\mathcal{I}_{|\Lambda|}$ for any $X \subset \Lambda$, since the coupling constants of $H^I_{\Lambda,\{\mathscr{J}_A(\theta)\}}$ are decreasing in θ. Analogously, $\langle V_X \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}$ is an increasing function on $\mathcal{I}_{|\Lambda|}$ for any $X \subset \Lambda$. Theorem 5.1 is then a simple consequence of Lemma 5.2, with $d\mu(\theta_x) = \frac{2}{\pi} d\theta_x$ and

$$p(\theta) = \frac{Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}{\int d\theta\, Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}. \tag{5.13}$$

The last step missing is to show that $p(\theta)$ defined as above fulfills hypothesis (5.8) of Lemma 5.2. This amounts to showing

$$\begin{aligned} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\vee\psi)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\wedge\psi)\}} &\geq Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\psi)\}}; && (5.14) \\ Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta\vee\psi)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta\wedge\psi)\}} &\geq Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\rm Is}_{\Lambda,\{\mathscr{J}_A(\psi)\}}. && (5.15) \end{aligned}$$

Since the arguments to prove these inequalities are very similar, we prove explicitly only the first one. Eq. (5.14) is equivalent to

$$\left(\frac{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\wedge\psi)\}}} \right)^{-1} \left(\frac{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\vee\psi)\}}}{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\psi)\}}} \right) \geq 1 \tag{5.16}$$

Notice that

$$\left(\frac{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)\}}}{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\wedge\psi)\}}} \right)^{-1} \left(\frac{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\vee\psi)\}}}{Z^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\psi)\}}} \right) = \frac{\langle {\rm e}^{-H^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\vee\psi)-\mathscr{K}_A(\psi)\}}} \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\psi)\}}}{\langle {\rm e}^{-H^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)-\mathscr{K}_A(\theta\wedge\psi)\}}} \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\wedge\psi)\}}}.$$

Thanks to Lemma 5.3, the functions whose expectation value we are computing above fulfill the hypothesis of Lemma 5.1 and Corollary 5.2. Then, applying Lemma 5.3 and Corollary 5.2,

$$\begin{aligned} \langle {\rm e}^{-H^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\vee\psi)-\mathscr{K}_A(\psi)\}}} \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\psi)\}} &\geq \langle {\rm e}^{-H^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)-\mathscr{K}_A(\theta\wedge\psi)\}}} \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\psi)\}} \\ &\geq \langle {\rm e}^{-H^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta)-\mathscr{K}_A(\theta\wedge\psi)\}}} \rangle^{\rm Is}_{\Lambda,\{\mathscr{K}_A(\theta\wedge\psi)\}}. \end{aligned} \tag{5.17}$$

Hence $p(\theta)$ has the required property. □

Proof of Theorem 5.2

Let us now turn to Theorem 5.2. In order to prove it, we need some preliminary results. We follow the framework described in [32, 60].

Lemma 5.4. *Let H_Λ^{cl} be the hamiltonian defined in* (5.3). *If $J_A^1 \geq |J_A^2|$ for all $A \subset \Lambda$ and $J_A^2 = 0$ for $|A|$ odd, then there exist non negative coupling constants $\{K_M\}_{M\in\mathbb{Z}^\Lambda}$ such that*

$$H_\Lambda^{\text{cl}}(\phi) = -\sum_{M\in\mathbb{Z}^\Lambda} K_M \cos\left(M\cdot\phi\right), \tag{5.18}$$

where, given $M \in \mathbb{Z}^\Lambda$, $M = (m_1, m_2, \ldots, m_\Lambda)$, $M\cdot\phi = \sum_{x\in\Lambda} m_x\phi_x$.

Proof of Lemma 5.4. The statement follows from the following two identities:

$$\cos(\theta)\cos(\chi) = \frac{1}{2}(\cos(\theta-\chi)+\cos(\theta+\chi)), \tag{5.19}$$

$$\sin(\theta)\sin(\chi) = \frac{1}{2}(\cos(\theta-\chi)-\cos(\theta+\chi)), \tag{5.20}$$

$\forall\theta,\chi\in[0,2\pi]$. □

A necessary step for Theorem 5.2 is duplication of variables [32]: we consider two sets of angles (i.e. spins) on the lattice instead of just one, and denote them by $\{\phi_x\}_{x\in\Lambda}$ and $\{\bar\phi_x\}_{x\in\Lambda}$. The hamiltonian for the $\{\bar\phi_x\}$ is

$$\begin{aligned}\bar H_\Lambda^{\text{cl}}(\bar\phi) &= -\sum_{A\subset\Lambda}\left(\bar J_A^1\bar\sigma_A^1 + \bar J_A^2\bar\sigma_A^2\right)\\ &= -\sum_{M\in\mathbb{Z}^\Lambda}\bar K_M\cos(M\cdot\bar\phi).\end{aligned} \tag{5.21}$$

Here, $\{\bar\sigma_x\}$ are related to $\{\bar\phi_x\}$ as in Eq.s (5.1) and (5.2). The $\bar J_A^i$ are non negative coupling constants with $\bar J_A^1 \geq |\bar J_A^2| \geq 0$ and $\{\bar K_M\}$ are as in Lemma 5.4. A composite hamiltonian can be defined as

$$\begin{aligned}&-\tilde H_\Lambda(\phi,\bar\phi) = -H_\Lambda^{\text{cl}}(\phi) - \bar H_\Lambda^{\text{cl}}(\bar\phi)\\ &= \sum_{M\in\mathbb{Z}^\Lambda}\tfrac{K_M+\bar K_M}{2}\left(\cos(M\cdot\phi)+\cos(M\cdot\bar\phi)\right) + \tfrac{K_M-\bar K_M}{2}\left(\cos(M\cdot\phi)-\cos(M\cdot\bar\phi)\right)\end{aligned} \tag{5.22}$$

In the following we always suppose $K_M \geq \bar K_M$ for all $M \in \mathbb{Z}^\Lambda$.The expectation value of any functional $f(\phi,\bar\phi)$ can be written as

$$\langle f\tilde{\rangle}_{\Lambda,\beta} = \frac{1}{Z^{\rm cl}_{\Lambda,\beta}\bar Z^{\rm cl}_{\Lambda,\beta}}\int d\phi\, d\bar\phi\, \mathrm{e}^{-\beta\tilde H_\Lambda(\phi,\bar\phi)} f(\phi,\bar\phi), \tag{5.23}$$

with $\bar Z^{\rm cl}_{\Lambda,\beta}$ the usual partition function with coupling constants $\{\bar J^i_A\}_{A\subset\Lambda}$, $i\in\{1,2\}$.

Lemma 5.5. *Suppose $f(\phi,\bar\phi)$ belongs to the cone generated by $\cos(M\cdot\phi)\pm\cos(M\cdot\bar\phi)$, $M\in\mathbb{Z}^\Lambda$, i.e. f can be written as product, sum or multiplication by a positive scalar of objects of that form. Then*

$$\langle f\tilde{\rangle}_{\Lambda,\beta}\geq 0. \tag{5.24}$$

Proof of Lemma 5.5. Firstly, notice that

$$\int d\phi\, d\bar\phi \prod_{s=1}^{n}\big(\cos(M_s\cdot\phi)\pm\cos(M_s\cdot\bar\phi)\big)\geq 0. \tag{5.25}$$

for any $M_1,\ldots,M_n\in\mathbb{Z}^\Lambda$ and any sequence of $(\pm)$. This follows from

$$\cos(M\cdot\phi)+\cos(M\cdot\bar\phi) = 2\cos(M\cdot\Phi)\cos(M\cdot\bar\Phi), \tag{5.26}$$

$$\cos(M\cdot\phi)-\cos(M\cdot\bar\phi) = 2\sin(M\cdot\Phi)\sin(M\cdot\bar\Phi), \tag{5.27}$$

with $\Phi_i = \frac12(\phi_i+\bar\phi_i)$ and $\bar\Phi_i = \frac12(\phi_i-\bar\phi_i)$. The integral in Eq. (5.25) can be formulated as

$$\int d\Phi\, d\bar\Phi F(\Phi)F(\bar\Phi) = \left(\int d\Phi F(\Phi)\right)^2\geq 0, \tag{5.28}$$

with $F(\Phi)$ an appropriate product of sines, cosines and positive constants.

Let us now turn to $\langle f\tilde{\rangle}_{\Lambda,\beta}$. Since the partition function is always positive, we can focus on

$$\int d\phi\, d\bar\phi\, \mathrm{e}^{-\beta\tilde H_\Lambda(\phi,\bar\phi)} f(\phi,\bar\phi). \tag{5.29}$$

By a Taylor expansion of $\mathrm{e}^{-\beta\tilde H_\Lambda(\phi,\bar\phi)}$ and by the properties of f, this can be expressed as a sum with positive coefficients of integrals in the form (5.25). Hence the nonnegativity of the expectation value. □

We have now all we need to prove Theorem 5.2.

Proof of Theorem 5.2. In order to prove the first statement of the theorem we use the formulation of the hamiltonian decribed in Lemma 5.4. Moreover, since σ^1_A can be clearly expressed as the sum (with positive coefficients) of terms of the form $\cos(M\cdot\phi)$, $M\in\mathbb{Z}^\Lambda$, it is enough to prove that for any $M,N\in\mathbb{Z}^\Lambda$

$$\begin{aligned}&\frac{\partial}{\partial K_N}\langle\cos(M\cdot\phi)\rangle^{\mathrm{cl}}_{\Lambda,\beta}\\&=\langle\cos(M\cdot\phi)\cos(N\cdot\phi)\rangle^{\mathrm{cl}}_{\Lambda,\beta}-\langle\cos(M\cdot\phi)\rangle^{\mathrm{cl}}_{\Lambda,\beta}\langle\cos(N\cdot\phi)\rangle^{\mathrm{cl}}_{\Lambda,\beta}\geq 0.\end{aligned}\tag{5.30}$$

Consider now the hamiltonian $\tilde{H}_\Lambda$ introduced above and $\langle\cdot\tilde{\rangle}_{\Lambda,\beta}$ the corresponding Gibbs state. From Lemma 5.5 we have

$$\langle\big(\cos(M\cdot\phi)-\cos(M\cdot\bar{\phi})\big)\big(\cos(N\cdot\phi)-\cos(N\cdot\bar{\phi})\big)\tilde{\rangle}_{\Lambda,\beta}\geq 0.\tag{5.31}$$

If we take the limit $\bar{K}_M\nearrow K_M$, we find twice the expression in Eq. (5.30). Hence the result.

Let us now turn to the second statement of the theorem. In the case of two-body interaction H^{cl}_Λ assumes the form in Eq. (5.5), which, with a notation resembling the one introduced in Lemma 5.4 can be explicitly formulated as

$$H^{\mathrm{cl}}_\Lambda(\phi)=-\sum_{x,y\in\Lambda}K^-_{xy}\cos(\phi_x-\phi_y)+K^+_{xy}\cos(\phi_x+\phi_y)\tag{5.32}$$

with

$$K^\pm_{xy}=\frac{J_{xy}}{2}\left(1\mp\eta_{xy}\right).\tag{5.33}$$

Cleary $K^\pm_{xy}$ is analogous to the K_M introduced in Lemma 5.4 for $M\in\mathbb{Z}^\Lambda$ such that all its elements are zero except $m_x=1$, $m_y=\pm1$. Then we have

$$\frac{\partial}{\partial J_{xy}}\langle\sigma_A\rangle^{\mathrm{cl}}_{\Lambda,\beta}=\frac{1+\eta_{xy}}{2}\frac{\partial}{\partial K^-_{xy}}\langle\sigma_A\rangle^{\mathrm{cl}}_{\Lambda,\beta}+\frac{1-\eta_{xy}}{2}\frac{\partial}{\partial K^+_{xy}}\langle\sigma_A\rangle^{\mathrm{cl}}_{\Lambda,\beta}.\tag{5.34}$$

Due to Eq. (5.30) the expression above is the sum of two nonnegative terms, hence it is nonnegative.

□

Proof of Theorem 5.3

Now we discuss the proof of Theorem 5.3. We use some of the concepts introduced in the proof of Theorem 5.1. The present proof has been proposed in [48, 47].

Proof of Theorem 5.3. As for the proof of Theorem 5.1, we express the XY spins by means of two Ising spins and an angle in $\left[0,\frac{\pi}{2}\right]$ - see Eq.s (5.9), (5.10) for the explicit expression of the spins and (5.12) for the new formulation of the hamiltonian H^{cl}_Λ.

With the same notation:

$$
\begin{aligned}
\langle \sigma_X^1 \rangle_\Lambda^{\mathrm{cl}} &= \frac{\int d\theta \, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}} \cos(\theta)_X \langle U_X \rangle^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}}}{\int d\theta \, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}}} \\
&\leq \frac{\int d\theta \, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}} \max_{\theta \in \mathcal{I}_{|\Lambda|}} \langle U_X \rangle^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}}}{\int d\theta \, Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{J}_A(\theta)\}} Z^{\mathrm{Is}}_{\Lambda,\{\mathscr{K}_A(\theta)\}}} \\
&= \langle U_A \rangle^{\mathrm{Is}}_{\Lambda,\{J_A^1\}}.
\end{aligned} \tag{5.35}
$$

□

5.2 Correlation inequalities for the quantum XY model

We have briefly introduced the quantum XY model in Ex. 3.3. We are interested here in a more general setting with many-body interactions. Let Λ be the set of sites hosting the quantum particles with spin $s \in \frac{1}{2}\mathbb{N}$. We focus on the cases $s = \frac{1}{2}$, 1. The Hilbert space of the model is naturally defined as $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2s+1}$. The hamiltonian under study is the following:

$$
H_\Lambda^{\mathrm{qu}} = -\sum_{A\subset\Lambda}\Big(J_A^1 \prod_{x\in A} \mathcal{S}_x^1 + J_A^2 \prod_{x\in A}\mathcal{S}_x^2 \Big). \tag{5.36}
$$

Here $J_A^i \geq 0$ for any $A \subset \Lambda$ and any $i \in \{1,2\}$, $\mathcal{S}^i$ ($i \in \{1,2,3\}$) are the spin-s operators on $\mathbb{C}^{2s+1}$. The Gibbs state is defined in the usual way: for any observable a

$$
\langle a \rangle^{\mathrm{qu}}_{\Lambda,\beta} = \frac{1}{Z^{\mathrm{qu}}_{\Lambda,\beta}} \mathrm{Tr}\; a\mathrm{e}^{-\beta H_\Lambda}, \qquad Z^{\mathrm{qu}}_{\Lambda,\beta} = \mathrm{Tr}\; \mathrm{e}^{-\beta H_\Lambda}. \tag{5.37}
$$

We also consider the so called Schwinger functions: for any $t \in [0,1]$ and any pair of observables a and b

$$
\langle a; b \rangle_{\Lambda,\beta}(t) = \frac{1}{Z^{\mathrm{qu}}_{\Lambda,\beta}} \mathrm{Tr}\; a\, \mathrm{e}^{-t\beta H^{\mathrm{qu}}_\Lambda} b\, \mathrm{e}^{-(1-t)\beta H^{\mathrm{qu}}_\Lambda}. \tag{5.38}
$$

We can now state generalised correlation inequalities for the spin-$\frac{1}{2}$ case:

Theorem 5.4 (Griffiths inequalities for spin $\mathbf{s} = \frac{\mathbf{1}}{\mathbf{2}}$)**.** *Let $J_A^i \geq 0$ for any $A \subset \Lambda$ and any $i \in \{1,2\}$. Then for any $A, B \subset \Lambda$, and any $t \in [0,1]$, we have*

$$
\begin{aligned}
&\Big\langle \prod_{x\in A} \mathcal{S}_x^1; \prod_{x\in B} \mathcal{S}_x^1 \Big\rangle_{\Lambda,\beta}(t) - \Big\langle \prod_{x\in A} \mathcal{S}_x^1 \Big\rangle^{\mathrm{qu}}_{\Lambda,\beta} \Big\langle \prod_{x\in B} \mathcal{S}_x^1 \Big\rangle^{\mathrm{qu}}_{\Lambda,\beta} \geq 0; \\
&\Big\langle \prod_{x\in A} \mathcal{S}_x^1; \prod_{x\in B} \mathcal{S}_x^2 \Big\rangle_{\Lambda,\beta}(t) - \Big\langle \prod_{x\in A} \mathcal{S}_x^1 \Big\rangle^{\mathrm{qu}}_{\Lambda,\beta} \Big\langle \prod_{x\in B} \mathcal{S}_x^2 \Big\rangle^{\mathrm{qu}}_{\Lambda,\beta} \leq 0.
\end{aligned}
$$

Notice that Griffiths inequalities of the form seen in Theorem 5.1 for the classical case are recovered when $t = 0, 1$. The proof of Theorem 5.4 can be found in Section 5.2.1, where we follow our work [9]. It has been proved independently firstly in a less general setting in [31] and subsequently in this more general framework in [76, 67, 9].

Interestingly this result appears to be not trivially true for the quantum Heisenberg ferromagnet. Indeed a toy version of the fully SU(2) invariant model has been provided explicitly, for which this result does not hold (nearest neighbours interaction on a three-sites chain with open boundary conditions) [38]. The question whether this result might still be established in a proper setting is still open.

This theorem has a straightforward corollary, i.e. monotonicity of correlation functions.

Corollary 5.3. *Under the same assumptions as in the above theorem, we have for all $A, B \subset \Lambda$ that*

$$\frac{\partial}{\partial J^1_A} \Big\langle \prod_{x \in B} \mathcal{S}^1_x \Big\rangle^{\text{qu}}_{\Lambda,\beta} \geq 0;$$
$$\frac{\partial}{\partial J^1_A} \Big\langle \prod_{x \in B} \mathcal{S}^2_x \Big\rangle^{\text{qu}}_{\Lambda,\beta} \leq 0.$$

Notice that there is no quantum version of Theorem 5.2, which provides monotonicity of classical correlation functions with respect to β. In the case of the spin-1 XY model, we provide a weaker version of Theorem 5.4.

Theorem 5.5 (Griffiths inequalities for spin $\mathbf{s = 1}$)**.** *Let $J^i_A \geq 0$ for any $A \subset \Lambda$ and any $i \in \{1, 2\}$. Then for any $A, B \subset \Lambda$, and any $t \in [0, 1]$, we have*

$$\lim_{\beta \to \infty} \Big[\Big\langle \prod_{x \in A} \mathcal{S}^1_x ; \prod_{x \in B} \mathcal{S}^1_x \Big\rangle_{\Lambda,\beta}(t) - \Big\langle \prod_{x \in A} \mathcal{S}^1_x \Big\rangle^{\text{qu}}_{\Lambda,\beta} \Big\langle \prod_{x \in B} \mathcal{S}^1_x \Big\rangle^{\text{qu}}_{\Lambda,\beta} \Big] \geq 0;$$
$$\lim_{\beta \to \infty} \Big[\Big\langle \prod_{x \in A} \mathcal{S}^1_x ; \prod_{x \in B} \mathcal{S}^2_x \Big\rangle_{\Lambda,\beta}(t) - \Big\langle \prod_{x \in A} \mathcal{S}^1_x \Big\rangle^{\text{qu}}_{\Lambda,\beta} \Big\langle \prod_{x \in B} \mathcal{S}^2_x \Big\rangle^{\text{qu}}_{\Lambda,\beta} \Big] \leq 0.$$

Notice that this theorem is restricted to the case of zero temperature, i.e. to the ground state of the model.

We conclude this section by remarking that correlation inequalities in the quantum case can be applied also to other models of interest. For example, we consider a certain formulation of Kitaev's model (see [44] for its original formulation and [5] for a review of the topic). Let $\Lambda \subset\subset \mathbb{Z}^2$ be a square lattice with edges $\mathcal{E}_\Lambda$. Each *edge* of the lattice hosts a spin-$\frac{1}{2}$ particle, i.e. the Hilbert space of this model is $\mathcal{H}_\Lambda^{\text{Kitaev}} = \otimes_{e\in\mathcal{E}_\Lambda}\mathbb{C}^2$. The Kitaev hamiltonian is

$$H_\Lambda^{\text{Kitaev}} = -\sum_{x\in\Lambda} J_x \prod_{\substack{e\in\mathcal{E}_\Lambda:\\ x\in e}} \mathcal{S}_e^1 - \sum_{F\subset\Lambda} J_F \prod_{e\subset F} \mathcal{S}_e^3, \tag{5.39}$$

where F denotes the faces of the lattice, i.e the unit squares which are the building blocks of $\mathbb{Z}^2$, J_x, J_F are ferromagnetic coupling constants and $\mathcal{S}_e^i = \mathcal{S}^i \otimes \mathbb{1}_{\mathcal{E}_\Lambda\setminus e}$. $H_\Lambda^{\text{Kitaev}}$ has the same structure as the hamiltonian in Eq. (5.36) so Ginibre inequalities apply as well. It is not clear, though, whether this might lead to useful results for the study of this specific model. Another relevant model is the *plaquette orbital model* that was studied in [82, 12]; interactions between neighbours x, y are of the form $-\mathcal{S}_x^i\mathcal{S}_y^i$, with i being equal to 1 or 3 depending on the edge.

5.2.1 Proofs for the $s = \frac{1}{2}$ case

We now discuss the proof of Theorem 5.4. This theorem has been proved for pair interactions in [31], and it has been proposed independently in various works for more generic interactions [76, 67, 9]. We describe here the simpler approach proposed in our work [9]. Since the temperature does not play any role from now on, we set $\beta = 1$ and omit any dependency on it in the following.

Notation. As for the classical case we introduce the notation $\mathcal{S}_A^i = \prod_{x\in A}\mathcal{S}_x^i$.

Recall the explicit form of spin-$\frac{1}{2}$ operators from Ex. 3.1. It is convenient to perform a unitary transformation on the hamiltonian in Eq. (5.36) and consider its version with interactions along the first and third directions of spin, namely

$$H_\Lambda^{\text{qu}} = -\sum_{A\subset\Lambda} J_A^1\mathcal{S}_A^1 + J_A^3\mathcal{S}_A^3, \tag{5.40}$$

with $J_A^3 = J_A^2$ for all $A \subset \Lambda$.

Proof of Theorem 5.4. The proof of this theorem uses some techniques similar to the ones introduced for the classical Theorem 5.2. These were indeed introduced by Ginibre [32] in a general framework. As for the classical case, it is convenient to duplicate the model. We introduce a new doubled Hilbert space $\bar{\mathcal{H}}_\Lambda = \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$. Given an operator $\mathcal{O}$ acting on $\mathcal{H}_\Lambda$ we define two operators acting on $\bar{\mathcal{H}}_\Lambda$,

$$\mathcal{O}_\pm = \mathcal{O}\otimes\mathbb{1} \pm \mathbb{1}\otimes\mathcal{O}. \tag{5.41}$$

The hamiltonian we consider for the doubled system is $H^{\mathrm{qu}}_{\Lambda,+}$:

$$H^{\mathrm{qu}}_{\Lambda,+} = H^{\mathrm{qu}}_{\Lambda} \otimes \mathbb{1}_{\Lambda} + \mathbb{1}_{\Lambda} \otimes H^{\mathrm{qu}}_{\Lambda} = -\sum_{A\subset\Lambda} J^1_A(\mathcal{S}^1_A)_+ + J^3_A(\mathcal{S}^3_A)_+ \tag{5.42}$$

The Gibbs state is denoted as

$$\langle\langle O\rangle\rangle_{\Lambda} = \frac{1}{(Z^{\mathrm{qu}}_{\Lambda})^2}\mathrm{Tr}\; O\mathrm{e}^{-H^{\mathrm{qu}}_{\Lambda,+}}, \tag{5.43}$$

for any operator O acting on $\bar{\mathcal{H}}_{\Lambda}$. Schwinger functions in this doubled setting are defined as

$$\langle\langle O;\, Q\rangle\rangle_{\Lambda}(t) = \frac{1}{(Z^{\mathrm{qu}}_{\Lambda})^2}\mathrm{Tr}\; O\,\mathrm{e}^{-tH^{\mathrm{qu}}_{\Lambda,+}}\, Q\,\mathrm{e}^{-(1-t)H^{\mathrm{qu}}_{\Lambda,+}} \tag{5.44}$$

for any pair of observables O and Q on $\bar{\mathcal{H}}_{\Lambda}$. It follows from some straightforward algebra that

$$\begin{aligned} \langle\mathcal{O};\mathcal{P}\rangle_{\Lambda}(t) - \langle\mathcal{O}\rangle^{\mathrm{qu}}_{\Lambda}\langle\mathcal{P}\rangle^{\mathrm{qu}}_{\Lambda} &= \frac{1}{2}\langle\langle\mathcal{O}_-;\mathcal{P}_-\rangle\rangle_{\Lambda}(t), && (5.45)\\ (\mathcal{O}\mathcal{P})_{\pm} &= \frac{1}{2}\left(\mathcal{O}_+\mathcal{P}_{\pm} + \mathcal{O}_-\mathcal{P}_{\mp}\right), && (5.46) \end{aligned}$$

for any $\mathcal{O}$, $\mathcal{P}$ operators on $\mathcal{H}_{\Lambda}$.

Just as $\mathbb{C}^2$ constitutes the "building block" for $\mathcal{H}_{\Lambda}$, so $\mathbb{C}^2\otimes\mathbb{C}^2$ is to $\bar{\mathcal{H}}_{\Lambda}$. We can provide an explicit basis of $\mathbb{C}^2\otimes\mathbb{C}^2$ such that $\mathcal{S}^1_+$, $\mathcal{S}^1_-$, $\mathcal{S}^3_+$, $-\mathcal{S}^3_-$ have all non negative elements:

$$\begin{aligned} |\psi_+\rangle &= \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right), \quad |\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle - |--\rangle\right), && (5.47)\\ |\chi_+\rangle &= \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right), \quad |\chi_-\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right). && (5.48) \end{aligned}$$

Above by $|+\rangle$ and $|-\rangle$ we denote the basis of $\mathbb{C}^2$ formed by eigenvectors of $\mathcal{S}^3$ with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$ respectively, and $|i,j\rangle = |i\rangle\otimes|j\rangle$. It can be easily checked that the basis above has the required property. This result implies straightforwardly that there exists a basis of $\bar{\mathcal{H}}_{\Lambda}$ such that $(\mathcal{S}^1_x)_+$, $(\mathcal{S}^1_x)_-$, $(\mathcal{S}^3_x)_+$ and $(-\mathcal{S}^3_x)_-$ have nonnegative element for all $x\in\Lambda$. Let us consider the truncated correlation function we are interested in:

$$\Big\langle\prod_{x\in X}\mathcal{S}^1_x;\prod_{x\in Y}\mathcal{S}^1_x\Big\rangle_{\Lambda}(t) - \Big\langle\prod_{x\in X}\mathcal{S}^1_x\Big\rangle^{\mathrm{qu}}_{\Lambda}\Big\langle\prod_{x\in Y}\mathcal{S}^1_x\Big\rangle^{\mathrm{qu}}_{\Lambda} = \tfrac{1}{2}\Big\langle\Big\langle\left(\mathcal{S}^1_X\right)_-\left(\mathcal{S}^1_Y\right)_-\Big\rangle\Big\rangle_{\Lambda}(t). \tag{5.49}$$

We can evaluate the right hand side of the equation above by a Taylor expansion:

$$(Z_\Lambda^{\mathrm{qu}})^2 \Big\langle\Big\langle (\mathcal{S}_X^1)_-; (\mathcal{S}_Y^1)_- \Big\rangle\Big\rangle_\Lambda (t) =$$

$$= \sum_{\substack{n\geq 0\\ m\geq 0}} \frac{t^n(1-t)^m}{n!m!} \mathrm{Tr}\ \left(\mathcal{S}_X^1\right)_- (-H_{\Lambda,+}^{\mathrm{qu}})^n \left(\mathcal{S}_Y^1\right)_- (-H_{\Lambda,+}^{\mathrm{qu}})^m \tag{5.50}$$

Given the formulation of $(-H_{\Lambda,+}^{\mathrm{qu}})$ as in Eq. (5.42) and the equality in Eq. (5.46), it is clear that it can be expressed as a polynomial with positive coefficients of operators with nonnegative elements. The same holds for $(\mathcal{S}_X^1)_-$ and $(\mathcal{S}_Y^1)_-$. The trace of operators with nonnegative elements is nonnegative, hence the first inequality of the theorem. The second inequality can be proved precisely in the same way (with $\mathcal{S}_Y^2$ substituted by $\mathcal{S}_Y^3$), by noticing that $(\mathcal{S}_Y^3)_-$ has necessarily nonpositive elements. $\square$

Proof of Corollary 5.3. This corollary is a straightforward application of Theorem 5.4. Indeed, it is enough to notice that

$$\frac{\partial}{\partial J_A^i} \langle \mathcal{S}_B^j \rangle_\Lambda^{\mathrm{qu}} = \int_0^1 dt\ \left(\langle \mathcal{S}_B^j; \mathcal{S}_A^i \rangle_\Lambda (t) - \langle \mathcal{S}_B^j \rangle_\Lambda^{\mathrm{qu}} \langle \mathcal{S}_A^i \rangle_\Lambda^{\mathrm{qu}} \right). \tag{5.51}$$

The result follows from Griffiths inequalities. $\square$

5.2.2 Proof for the $s = 1$ case

In this section we discuss the proof of Theorem 5.5, as proposed in our work [9]. We prove Griffiths inequalities at zero temperature only.

Our approach is as follows. We map the spin-1 model to a spin-$\frac{1}{2}$ model with two particles on each site, and use Theorem 5.4 to show correlation inequalities for the spin-1 model. Such a mapping holds only in the limit $\beta \to \infty$ thus restricting the result, though the proof is considerably trickier than the one for the spin-$\frac{1}{2}$ model. The idea is inspired by paper [61] about the loop representation of the quantum Heisenberg model. The proof is structured as follows:

1. We define a doubled model with two spin-$\frac{1}{2}$ particles per site instead of a spin-1 one.

2. We make it explicit that it is a spin-$\frac{1}{2}$ model on a peculiar lattice, so that correlation inequalities hold by Theorem 5.4.

3. We show that the ground state of this model is in some sense *effectively spin-1.*

4. We use correlation inequalities for this doubled model at the ground state to prove correlation inequalities for the spin-1 model at the ground state.

Notation. In this section we denote by $\mathcal{S}^i$ the spin-1 operators and by s^i the spin-$\frac{1}{2}$ ones. Moreover we denote by $\mathcal{S}^i_A = \prod_{x\in A} \mathcal{S}^i_x$ and $s^i_A = \prod_{x\in A} s^i_x$ for any set of sites A and for for any $i \in \{1,2,3\}$.

As for the spin-$\frac{1}{2}$ case, we prefer to work with the rotated Hamiltonian with interactions in the 1-3 directions,

$$H^{\mathrm{qu}}_\Lambda = -\sum_{A\subset\Lambda} J^1_A \mathcal{S}^1_A + J^3_A \mathcal{S}^3_A \tag{5.52}$$

with $J^3_A = J^2_A$ for any $A \subset \Lambda$.

Proof of Theorem 5.5. Firstly, let us introduce the doubled spin-$\frac{1}{2}$ model mentioned above. Let Λ be the physical space hosting the spin-1 particles, and denote $\hat\Lambda = \Lambda \times \{1,2\}$ its “doubled” version, i.e. to each site (particle) in Λ correspond two sites (particles) in $\hat\Lambda$. The Hilbert space is $\hat{\mathcal{H}}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \otimes_{x\in\hat\Lambda}\mathbb{C}^2$. We define the spin operators on a pair of spin-$\frac{1}{2}$ particles as the following operators on $\mathbb{C}^2\otimes\mathbb{C}^2$:

$$\mathcal{R}^i = s^i \otimes \mathbb{1} + \mathbb{1} \otimes s^i, \tag{5.53}$$

where $\mathbb{1}$ is the identity on $\mathbb{C}^2$. Notice that $\mathbb{C}^2\otimes\mathbb{C}^2$ can be decomposed into the orthogonal sum between the triplet space (with dimension three) and the singlet space (with dimension 1). We denote by $\mathcal{P}^{\mathrm{trip}}$ the projector on the triplet space. We define the isometry $\mathcal{V} : \mathbb{C}^3 \to \mathbb{C}^2\otimes\mathbb{C}^2$ such that

$$\begin{aligned} \mathcal{V}^*\mathcal{V} &= \mathbb{1}_{\mathbb{C}^3}, \\ \mathcal{V}\mathcal{V}^* &= \mathcal{P}^{\mathrm{trip}}. \end{aligned} \tag{5.54}$$

This isometry makes the relationship between spin-1 matrices $\mathcal{S}^i$ and the operators $\mathcal{R}^i$ explicit: $\mathcal{S}^i = \mathcal{V}^*\mathcal{R}^i\mathcal{V}$.

We define the following hamiltonian on $\hat{\mathcal{H}}_\Lambda$:

$$\hat H_\Lambda = -\sum_{A\subset\Lambda} J^1_A \prod_{x\in A} \mathcal{R}^1_x + J^3_A \prod_{x\in A} \mathcal{R}^3_x. \tag{5.55}$$

In the expression above $\mathcal{R}^i_x = \mathcal{R}^i \otimes \mathbb{1}_{\Lambda\setminus\{x\}}$ and the coupling constants $\{J^i_A\}_{A\subset\Lambda}$ are the same as in Eq. (5.52). The Gibbs state is defined as usual:

$$\widehat{\langle a \rangle}_{\Lambda,\beta} = \frac{1}{\hat{Z}_{\Lambda,\beta}} \mathrm{Tr}\; a \mathrm{e}^{-\beta \hat{H}_\Lambda} \quad \text{with } \hat{Z}_{\Lambda,\beta} = \mathrm{Tr}\; \mathrm{e}^{-\beta \hat{H}_\Lambda}. \tag{5.56}$$

for any operator a acting on $\hat{\mathcal{H}}_\Lambda$. Analogously, Schwinger functions are defined as

$$\widehat{\langle a; b \rangle}_{\Lambda,\beta}(t) = \frac{1}{\hat{Z}_{\Lambda,\beta}} \mathrm{Tr}\; a \mathrm{e}^{-t\beta \hat{H}_\Lambda} b \mathrm{e}^{-(1-t)\beta \hat{H}_\Lambda} \quad \text{with } t \in [0,1]. \tag{5.57}$$

It is clear that this model is a spin-$\frac{1}{2}$ system on a peculiar lattice. $\hat{H}_\Lambda$ can indeed be explicitly formulated as a spin-$\frac{1}{2}$ hamiltonian on the lattice $\hat{\Lambda}$:

$$\hat{H}_\Lambda = -\sum_{X \subset \hat{\Lambda}} \hat{J}^1_X s^1_X + \hat{J}^3_X s^3_X. \tag{5.58}$$

The coupling constants $\{\hat{J}^i_X\}_{X \subset \hat{\Lambda}}$ can be expressed non-trivially in terms of the original couplings $\{J^i_A\}_{A \subset \Lambda}$. Let D_Λ be defined as follows:

$$\begin{aligned} D_\Lambda = \{X \subset \hat{\Lambda} \,|\, \forall x \in \Lambda \text{ either } (x,i) \notin X \text{ for any } i \in \{1,2\} \\ \text{or } \exists! i \in \{1,2\} \text{ s.t. } (x,i) \in X\}, \end{aligned} \tag{5.59}$$

i.e. D_Λ is composed by subsets of $\hat{\Lambda}$ where the same site $x \in \Lambda$ does not appear in more than one copy. For any $X \subset \hat{\Lambda}$ define

$$\mathrm{supp(X)} = \{\mathrm{x} \in \Lambda \,|\, \exists\, \mathrm{i} \in \{1,2\} \text{ s.t. } (\mathrm{x},\mathrm{i}) \in \mathrm{X}\} \subset \Lambda. \tag{5.60}$$

Then the coupling constants $\{\hat{J}^i_X\}_{X \subset \hat{\Lambda}}$ have the following form:

$$\hat{J}^i_X = \begin{cases} J^i_{\mathrm{supp(X)}} & \text{if } X \in D_\Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{5.61}$$

This formulation of $\hat{H}_\Lambda$ assures us that Griffiths inequalities can be applied to this model thanks to Theorem 5.4.

We would like to show that the ground state of $\hat{H}_\Lambda$ lies in the triplet subspace of $\hat{\mathcal{H}}_\Lambda$ identified by the projector $\mathcal{P}^{\mathrm{trip}}_\Lambda = \otimes_{x \in \Lambda} \mathcal{P}^{\mathrm{trip}}_x$ (here $\mathcal{P}^{\mathrm{trip}}_x = \mathcal{P}^{\mathrm{trip}} \otimes \mathbb{1}_{\Lambda \setminus \{x\}}$). To do this we need two steps:

1. We first prove that the ground state energy of $\hat{H}_\Lambda$ is a strictly decreasing function of the couplings $\{J^i_A\}_{A \subset \Lambda}$.

2. We show how this implies that the ground state of $\hat{H}_\Lambda$ lies in the triplet subspace.

The first step is equivalent to proving that for any $Y \subset \Lambda$

$$E_0\left(\hat{H}_\Lambda - \epsilon \prod_{x\in Y} s_x^i\right) < E_0(\hat{H}_\Lambda), \tag{5.62}$$

where by $E_0(\cdot)$ we denote the lowest eigenvalue and $i \in \{1,3\}$. We focus on the case $i = 1$ only. Let Ψ_0 be the ground state of $\hat{H}_\Lambda$, and notice that it is also the eigenvector of the operator $\mathrm{e}^{-\hat{H}_\Lambda}$ with the highest eigenvalue. By the Trotter formula (see Appendix A)

$$\mathrm{e}^{-\hat{H}_\Lambda} = \lim_{k\to\infty}\left(\left(1 + \frac{1}{k}\sum_{X\subset\hat{\Lambda}} \hat{J}_X^1 s_X^1\right)\mathrm{e}^{\frac{1}{k}\sum_{X\subset\hat{\Lambda}} \hat{J}_X^3 s_x^3}\right)^k. \tag{5.63}$$

Notice that due to the explicit form of spin-$\frac{1}{2}$ matrices (Ex. 3.1) the formula above is the product of matrices with non-negative elements. This allows us to use a Perron-Frobenius argument, which implies that Ψ_0 in this basis is a vector with non-negative coefficients. Then

$$\begin{aligned} E_0\left(\hat{H}_\Lambda - \epsilon \prod_{x\in Y} s_x^1\right) &\le \left(\Psi_0, \left(\hat{H}_\Lambda - \epsilon \prod_{x\in Y} s_x^1\right)\Psi_0\right) \\ &= E_0(\hat{H}_\Lambda) - \epsilon\left(\Psi_0, \left(\prod_{x\in Y} s_x^1\right)\Psi_0\right). \end{aligned} \tag{5.64}$$

The result follows in case $\left(\Psi_0, \left(\prod_{x\in Y} s_x^1\right)\Psi_0\right) \neq 0$. Otherwise, we shift the hamiltonian: $\hat{H}'_\Lambda = \hat{H}_\Lambda - c\mathbb{1}$, where $c > 0$ is big enough so that $\hat{H}'_\Lambda$ has negative eigenvalues only. Since $(s^1)^2 = \frac{1}{4}\mathbb{1}_{\mathbb{C}^2}$

$$\left(\Psi_0, \left(\hat{H}'_\Lambda - \epsilon \prod_{x\in Y} s_x^1\right)^2 \Psi_0\right) = E_0(\hat{H}'_\Lambda)^2 + 4^{-|Y|}\epsilon^2. \tag{5.65}$$

This implies that

$$E_0\left(\hat{H}'_\Lambda - \epsilon \prod_{x\in Y} s_x^1\right) < E_0(\hat{H}'_\Lambda). \tag{5.66}$$

In this way we have proved that Eq.(5.62) holds for $i = 1$. The case $i = 3$ follows simply by a unitary transformation.

The fact that the ground state energy of $\hat{H}_\Lambda$ is a decreasing function of the coupling constants implies that its ground state lies in the triplet subspace. To show this, define the operator $\mathcal{Q}_{\Lambda,A} = \left(\otimes_{x\in A}\mathcal{P}_x^{\mathrm{trip}}\right) \otimes \left(\otimes_{x\in\Lambda\setminus A}(1-\mathcal{P}_x^{\mathrm{trip}})\right)$. Notice that $[\mathcal{R}_x^i, \mathcal{Q}_{\Lambda,A}] = 0$ for any $A \subset \Lambda$, $x \in \Lambda$ and $i \in \{1,2,3\}$. Recall the formulation of $\hat{H}_\Lambda$ in terms of the operators $\mathcal{R}^i$, Eq. (5.55). Notice that

$$\mathcal{Q}_{\Lambda,A}\hat{H}_\Lambda = -\mathcal{Q}_{\Lambda,A}\sum_{B\subset A}\left(J_B^1\prod_{x\in B}\mathcal{R}_x^1 + J_B^3\prod_{x\in B}\mathcal{R}_x^3\right). \tag{5.67}$$

This follows from the commutators above and from the fact that $\mathcal{R}^i$ applied to singlets gives zero. This implies that the ground state of $\hat{H}_\Lambda$ lies in the triplet state. Indeed $\mathcal{Q}_{\Lambda,A}$ projects into the triplet space over A and into the singlet space over $\Lambda\setminus A$. The equation above shows that its action is equivalent to setting the coupling constants outside of A equal to zero. Since the ground state energy is a decreasing function of the coupling constants and $\mathcal{Q}_{\Lambda,\Lambda} = \mathcal{P}_\Lambda^{\mathrm{trip}}$, it follows that the ground state lies in the triplet state.

This allows us to prove correlation inequalities for the spin-1 model in the ground state. Indeed by Theorem 5.4 we have:

$$\begin{aligned}
\lim_{\beta\to\infty}\left\langle\mathcal{S}_A^1;\mathcal{S}_B^1\right\rangle_{\Lambda,\beta}^{\mathrm{qu}}(t) &= \lim_{\beta\to\infty}\sum_{\substack{X\in D_\Lambda:\\ \mathrm{supp}(X)=A}}\sum_{\substack{Y\in D_\Lambda:\\ \mathrm{supp}(Y)=B}}\langle s_X^1;s_Y^1\hat{\rangle}_{\Lambda,\beta}(t)\\
&\geq \lim_{\beta\to\infty}\sum_{\substack{X\in D_\Lambda:\\ \mathrm{supp}(X)=A}}\sum_{\substack{Y\in D_\Lambda:\\ \mathrm{supp}(Y)=B}}\langle s_X^1\hat{\rangle}_{\Lambda,\beta}\langle s_Y^1\hat{\rangle}_{\Lambda,\beta}\\
&= \lim_{\beta\to\infty}\left\langle\mathcal{S}_A^1\right\rangle_{\Lambda,\beta}^{\mathrm{qu}}\left\langle\mathcal{S}_B^1\right\rangle_{\Lambda,\beta}^{\mathrm{qu}}.
\end{aligned} \tag{5.68}$$

The other inequality in Theorem 5.5 is proved in the same way.

□

5.3 Applications

In this section we analyse some potentially interesting applications of Ginibre inequalities for the spin-$\frac{1}{2}$ quantum XY model. Firstly, we use Corollary 5.3 to compare the critical temperature of quantum XY and classical Ising models, as shown in [76, 66]. Our discussion follows closely our review [10]. Secondly, we study the infinite volume limit of some correlation functions, adapting our discussion from [9]. Lastly, we focus on systems with a random coupling and we prove some new results about the relationship between the annealed and quenched case.

Notation. Since we are now interested only in the case $s = \frac{1}{2}$, throughout this section we denote the spin-$\frac{1}{2}$ matrices with $\mathcal{S}^i$, $i \in \{1, 2, 3\}$.

5.3.1 Comparison between Ising and XY model

Let the Ising model be defined as in Eq. (5.6). Then the following statement holds for spin $\frac{1}{2}$.

Theorem 5.6. *Assume that $J_A^1, J_A^2 \geq 0$ for all $A \subset \Lambda$. Then for all $X \subset \Lambda$:*

$$\left\langle \prod_{x\in X} \mathcal{S}_x^1 \right\rangle_{\Lambda,\beta}^{\text{qu}} \leq 2^{-|X|} \left\langle \prod_{x\in X} \omega_x \right\rangle_{\Lambda,\{J_A^*\},\beta}^{\text{Is}},$$

with $J_A^* = 2^{-|A|} J_A^1$.

The statement can be easily recovered by recalling that the *classical* Ising model can be recovered as a particular case of the *quantum* XY model (not of the classical one!) and by using Corollary 5.3. Interestingly this result has been extended to any value of the spin [66]. We define the *spontaneous magnetisation* $m^{\#}(\beta)$ at inverse temperature β by

$$m^{\text{Is}}(\beta)^2 = \liminf_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|^2} \sum_{x,y\in\Lambda} \langle \omega_x^1 \omega_y^1 \rangle_{\Lambda,\beta}^{\text{Is}}, \tag{5.69}$$

$$m^{\text{qu}}(\beta)^2 = \liminf_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|^2} \sum_{y\in\Lambda} \langle \mathcal{S}_x^1 \mathcal{S}_y^1 \rangle_{\Lambda,\beta}^{\text{qu}}. \tag{5.70}$$

We define the critical temperature $T_{\text{c}}^{\#} = 1/\beta_{\text{c}}^{\#}$ as

$$\beta_{\text{c}}^{\#} = \sup\{\beta > 0 : m^{\#}(\beta) = 0\}, \tag{5.71}$$

where $\beta_{\text{c}}^{\#} \in (0, \infty]$. For quantum XY and Ising models with pair interaction the next statement follows straightforwardly from Theorem 5.6.

Corollary 5.4. *The critical temperatures satisfy*

$$T_{\text{c}}^{\text{qu}} \leq \tfrac{1}{4} T_{\text{c}}^{\text{Is}}.$$

5.3.2 Infinite volume limit of correlation functions

In this section we provide a result from [9] potentially very useful in the study of the infinite volume Gibbs state for the spin-$\frac{1}{2}$ quantum XY model. As explained in Chapter 3, the definition and existence of the infinite volume Gibbs state for quantum models is highly non trivial. We show here that for the quantum XY model with $s = \frac{1}{2}$ the infinite volume limit of certain correlations is well defined for a suitable Gibbs state with *+-boundary conditions*. Though this result is only partial, we hope that this could a be first step in the definition of the infinite volume Gibbs state of this model.

Throughout this section we assume that the interaction has finite range (recall the definition form Chapter 3), and we denote by R the range of the interaction. The inverse temperature β is set to be equal to 1 since it does not play any role, and we drop any dependency on it in the notation.

Our first step is to introduce a well defined finite volume Gibbs state which describes +-boundary conditions. In order to do so, we enlarge our lattice Λ by R and define $\Lambda_R = \{x \in \mathbb{Z}^d : \mathrm{d}(x, \Lambda) \leq R\}$. The external layer of Λ_R is defined as $\partial_R \Lambda = \Lambda_R \backslash \Lambda$. We set a positive magnetic field η in the 1-direction in $\partial_R \Lambda$. The new hamiltonian is then the following operator acting on $\mathcal{H}_{\Lambda_R} = \otimes_{x \in \Lambda_R} \mathbb{C}^2$

$$H^\eta_{\Lambda_R} = - \sum_{A \subset \Lambda_R} \left(J^1_A \prod_{x \in A} \mathcal{S}^1_x + J^2_A \prod_{x \in A} \mathcal{S}^2_x \right) - \eta \sum_{x \in \partial_R \Lambda} \mathcal{S}^1_x. \tag{5.72}$$

The new Gibbs state is defined as

$$\langle a \rangle^\eta_{\Lambda_R} = \frac{1}{Z^\eta_{\Lambda_R}} \mathrm{Tr}\ a\, \mathrm{e}^{-H^\eta_{\Lambda_R}} \quad \text{with } Z^\eta_{\Lambda_R} = \mathrm{Tr}\ \mathrm{e}^{-H^\eta_{\Lambda_R}} \tag{5.73}$$

for any observable a. For an infinite magnetic field, this should describe the finite volume Gibbs state with +-boundary conditions in the first spin direction. The following theorem states that such a limit exists.

Proposition 5.1 (Finite volume Gibbs state with +-boundary conditions)**.** *Let a be an observable acting on $\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^2$. Then the limit $\lim_{\eta \to \infty} \langle a \rangle^\eta_{\Lambda_R}$ exists – where a has been identified with $a \otimes \mathbb{1}_{\partial_R \Lambda}$ acting on $\mathcal{H}_{\Lambda_R}$. We denote the limit with $\langle a \rangle^+_\Lambda$. Moreover:*

$$\langle a \rangle^+_\Lambda = \frac{Tr\ a e^{-H^+_\Lambda}}{Tr\ e^{-H^+_\Lambda}},$$

where the traces are on $\mathcal{H}_\Lambda$ and H^+_Λ is the following operator on $\mathcal{H}_\Lambda$:

$$H^+_\Lambda = - \sum_{A \subset \Lambda} J^1_A \prod_{x \in A} \mathcal{S}^1_x + J^3_A \prod_{x \in A} \mathcal{S}^3_x - \sum_{\substack{A \subset \Lambda_R: \\ A \cap \Lambda_R \neq 0}} 2^{-|A \cap \partial_R \Lambda|} J^1_A \prod_{x \in A \cap \Lambda} \mathcal{S}^1_x.$$

Proof. Let us shift $H^{\eta}_{\Lambda_R}$ by a constant, since this does not affect the Gibbs state:

$$\langle a\rangle^{\eta}_{\Lambda} = \frac{\mathrm{Tr}\ a\mathrm{e}^{-H^{\eta}_{\Lambda_R}-\frac{\eta}{2}|\partial_R\Lambda|}}{\mathrm{Tr}\ \mathrm{e}^{-H^{\eta}_{\Lambda_R}-\frac{\eta}{2}|\partial_R\Lambda|}}. \tag{5.74}$$

We can apply Trotter formula (see Appendix A) and find

$$\begin{aligned}&\lim_{\eta\to\infty} \mathrm{Tr}\ a\mathrm{e}^{-H^{\eta}_{\Lambda_R}-\frac{\eta}{2}|\partial\Lambda_R|}\\ &= \lim_{\eta\to\infty}\lim_{k\to\infty} \mathrm{Tr}\ a \left(\left(1+\frac{1}{k}\sum_{A\subset\Lambda_R}\left(J^1_A\mathcal{S}^1_A + J^2_A\mathcal{S}^2_A\right)\right)\mathrm{e}^{\frac{\eta}{k}\sum_{x\in\partial_R\Lambda}\left(\mathcal{S}^1_x-\frac{1}{2}\right)}\right)^k.\end{aligned} \tag{5.75}$$

Notice that we can invert the order of the limits and define the projector P_x for any $x\in\Lambda$:

$$P_x = \lim_{\eta\to\infty} \mathrm{e}^{\eta(\mathcal{S}^1_x-\frac{1}{2})}. \tag{5.76}$$

It is straightforward to check that this operator projects on the the eigenstate with eigenvalue $\frac{1}{2}$ of the spin matrix $\mathcal{S}^1$ at site $x\in\Lambda$. Then

$$\lim_{\eta\to\infty} \mathrm{Tr}\ a\mathrm{e}^{-H^{\eta}_{\Lambda_R}-\frac{\eta}{2}|\partial\Lambda_R|} = \lim_{k\to\infty} \mathrm{Tr}\ a \left(\left(1+\frac{1}{k}\sum_{A\subset\Lambda_R} J^1_A\mathcal{S}^1_A + J^2_A\mathcal{S}^2_A\right)P_{\partial_R\Lambda}\right)^k, \tag{5.77}$$

where we have introduced the notation $P_{\partial_R\Lambda} = \prod_{x\in\partial_R\Lambda} P_x$. Notice that

$$\begin{aligned}&P_{\partial_R\Lambda}\mathcal{S}^1_A P_{\partial_R\Lambda} = 2^{-|A\cap\partial_R\Lambda|}\mathcal{S}^1_{A\cap\Lambda}P_{\partial_R\Lambda},\\ &P_{\partial_R\Lambda}\mathcal{S}^2_A P_{\partial_R\Lambda} = 0 \quad \text{if } A\cap\partial_R\Lambda \neq \emptyset.\end{aligned} \tag{5.78}$$

This implies that

$$\lim_{\eta\to\infty} \mathrm{Tr}\ a\,\mathrm{e}^{-H^{\eta}_{\Lambda_R}-\frac{\eta}{2}|\partial\Lambda_R|} = \lim_{k\to\infty} \mathrm{Tr}\ a\left(1-\frac{1}{k}H^{+}_{\Lambda}\right)^k = \mathrm{Tr}\ a\,\mathrm{e}^{-H^{+}_{\Lambda}}. \tag{5.79}$$

□

Notice that if $J^1_A = J^2_A$ for any $A\subset\Lambda$ (i.e. for the general XX model), the infinite volume limit of the state $\langle\cdot\rangle^{+}_{\Lambda}$ at low temperature is expected to be extremal and describe the state with magnetisation along the first direction of spin. Though the existence of the infinite volume limit is far from being proved, we now provide a partial result in this direction, namely the existence of certain infinite volume correlation functions.

Theorem 5.7. *The limit* $\lim_{\Lambda\nearrow\mathbb{Z}^d}\langle\mathcal{S}_A^i\rangle_\Lambda^+$ *exists and is finite for any* $i\in\{1,2\}$ *and* $A\subset\Lambda$.

Proof. We can generalise the setting described so far and for any $\Lambda'\supset\Lambda$ we can define

$$H^\eta_{\Lambda,\Lambda'}=-\sum_{A\subset\Lambda'_R}\left(J_A^1\mathcal{S}_A^1+J_A^2\mathcal{S}_A^2\right)-\eta\sum_{x\in\Lambda'_R\setminus\Lambda}\mathcal{S}_x^1. \tag{5.80}$$

This hamiltonian acts on $\mathcal{H}_{\Lambda'_R}$. The idea is precisely the same one as discussed above, and it is straightforward to check that:

$$\langle a\rangle_\Lambda^+=\lim_{\eta\to\infty}\frac{\mathrm{Tr}\ a\,\mathrm{e}^{-H^\eta_{\Lambda,\Lambda'}}}{\mathrm{Tr}\ \mathrm{e}^{-H^\eta_{\Lambda,\Lambda'}}}. \tag{5.81}$$

Notice that by Corollary 5.3 we have that

$$\frac{\mathrm{Tr}\ \mathcal{S}_A^1\,\mathrm{e}^{-H^\eta_{\Lambda,\Lambda'_R}}}{\mathrm{Tr}\ \mathrm{e}^{-H^\eta_{\Lambda,\Lambda'_R}}}\geq\frac{\mathrm{Tr}\ \mathcal{S}_A^1\,\mathrm{e}^{-H^\eta_{\Lambda'_R}}}{\mathrm{Tr}\ \mathrm{e}^{-H^\eta_{\Lambda'_R}}} \tag{5.82}$$

By taking the limit $\eta\to\infty$ for both sides of the inequality, we have

$$\langle\mathcal{S}_A^1\rangle_\Lambda^+\geq\langle\mathcal{S}_A^1\rangle_{\Lambda'}^+. \tag{5.83}$$

$\langle\mathcal{S}_A^1\rangle_\Lambda^+$ is thus a bounded nonincreasing sequence, so the limit exists and is finite. The same result can be proved for $\langle\mathcal{S}_A^2\rangle_\Lambda^+$, with the only difference that by Corollary 5.3 this sequence is nondecreasing instead of nonincreasing. □

5.3.3 Quenched and annealed averages in the quantum XY model

An interesting problem is the behaviour of lattice spin systems with some sort of disorder. The idea is to study what happens to usual statistical mechanical models when some coupling constants or the magnetic field do not take a fixed value anymore, but are considered to be random variables (see e.g. the reviews [58, 15]). There are two possible ways of considering averages in this random setting, the so called *annealed* and *quenched* averages. In the first case, the average with respect to the disorder is taken before the thermodynamic one. In the second case, we first freeze the disorder and calculate the Gibbs state with it, and the average with respect to the randomness is then calculated afterwards. From a physical point of view, the quenched average is the most interesting one, but it is also the most complicated to handle. The annealed averaging on the other hand retains some appeal thanks to its simplicity.

In this section we are interested in the relationship between quenched and annealed averages for the quantum XY model with random coupling constants along one of the two directions of spin. In particular we focus on correlators for the quantum spin-$\frac{1}{2}$ XY model on a lattice Λ with hamiltonian $H^{\mathrm{qu}}_{\Lambda}$ defined in Eq. (5.36).

Assume that the coupling constants $\{J^1_X\}_{X\subset\Lambda}$ are non-negative and independently (but not necessarily identically!) distributed random variables. For any $X \subset \Lambda$ we denote with ν_X the probability distribution of J^1_X. Let J^1 denote the set of all coupling constants along the first axis. Then for any function of these coupling constants, $f(J^1)$, the average of f with respect to the probability distributions of the $\{J^1_X\}_{X\subset\Lambda}$ is denoted by

$$\mathbb{E}_{J^1}(f) = \int \prod_{X\subset\Lambda} d\nu_X(J^1_X) f(J^1). \tag{5.84}$$

We now turn to quenched and annealed averages. Given any observable a on $\mathcal{H}_\Lambda$, they are defined respectively as

$$\langle a\rangle^A_{\Lambda,\beta} = \frac{1}{\mathbb{E}_{J^1}\left(Z^{\mathrm{qu}}_{\Lambda,\beta}\right)}\mathbb{E}_{J^1}(\mathrm{Tr}\ a\,\mathrm{e}^{-\beta H^{\mathrm{qu}}_{\Lambda}}) = \frac{1}{\mathbb{E}_{J^1}\left(Z^{\mathrm{qu}}_{\Lambda,\beta}\right)}\mathbb{E}_{J^1}\left(Z^{\mathrm{qu}}_{\Lambda,\beta}\langle a\rangle^{\mathrm{qu}}_{\Lambda,\beta}\right), \tag{5.85}$$

$$\langle a\rangle^Q_{\Lambda,\beta} = \mathbb{E}_{J^1}\left(\frac{1}{Z^{\mathrm{qu}}_{\Lambda,\beta}}\mathrm{Tr}\ a\,\mathrm{e}^{-\beta H^{\mathrm{qu}}_{\Lambda}}\right) = \mathbb{E}_{J^1}\left(\langle a\rangle^{\mathrm{qu}}_{\Lambda,\beta}\right). \tag{5.86}$$

We are able to provide an inequality between this two quantities, as stated in the following theorem.

Theorem 5.8. *Let H_Λ be the Hamiltonian of the quantum XY model with spin-$\frac{1}{2}$ on a lattice $\Lambda \subset\subset \mathbb{Z}^d$, as defined in Eq. (5.36). If the coupling constants along the first axis $\{J^1_X\}_{X\subset\Lambda}$ are non negative indipendently distributed random variables, then*

$$\left\langle \prod_{x\in A} \mathcal{S}^1_x \right\rangle^A_{\Lambda,\beta} \geq \left\langle \prod_{x\in A} \mathcal{S}^1_x \right\rangle^Q_{\Lambda,\beta} \quad \forall A\subset\Lambda, \tag{5.87}$$

$$\left\langle \prod_{x\in A} \mathcal{S}^2_x \right\rangle^A_{\Lambda,\beta} \leq \left\langle \prod_{x\in A} \mathcal{S}^2_x \right\rangle^Q_{\Lambda,\beta} \quad \forall A\subset\Lambda. \tag{5.88}$$

A similar result was proved in [72] for the nearest neighbours Ising model with random couplings. Before turning to turning to the proof of the theorem, we introduce a necessary lemma, the so called *Harris inequality.* This very well known result takes its name from the mathematician who first introduced it in his study of percolation [37]. We introduce it here in a more general context (see for example [72]).

Lemma 5.6 (Harris inequality)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$. Let f, g be monotonic in each of their variable, and assume that, for any $i \in \{1, \ldots, n\}$, f and g are both nondecreasing (or nonincreasing) in x_i. Let $p(x_1, \ldots, x_n)$ be a factorisable probability distribution: $p(x_1, \ldots, x_n) = \prod_{k=1}^n p_k(x_k)$. Then, with respect to this distribution*

$$\mathbb{E}(fg) \geq \mathbb{E}(f)\mathbb{E}(g).$$

Proof of Lemma 5.6. The proof goes by induction on the number of variables (n). Consider the case $n = 1$, that is, f and g depend on only one variable. Then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} [f(x) - f(y)] [g(x) - g(y)] \, p(x)p(y)dxdy \geq 0, \tag{5.89}$$

due to the monotonicity property of f and g, which implies that $[f(x) - f(y)]$ and $[g(x) - g(y)]$ have the same sign for any $x, y \in \mathbb{R}$. The inequality for $n = 1$ follows with some algebra. Suppose now that the statement holds for $n = k - 1$ for some $k \in \mathbb{N}$. We want to prove that it implies it holds for $n = k$. We have:

$$\begin{aligned}
&\int_{\mathbb{R}^k} f(x_1, \ldots, x_k) g(x_1, \ldots, x_k) \prod_{l=1}^{k} p_l(x_l) dx_l \\
&\geq \int_{\mathbb{R}^{k-1}} \prod_{l=1}^{k-1} p_l(x_l) dx_l \left(\int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, \ldots, x_{k-1}, x) g(x_1, \ldots, x_{k-1}, y) p_k(x) p_k(y) dx dy \right) \\
&\geq \left(\int_{\mathbb{R}^k} f(x_1, \ldots, x_k) \prod_{l=1}^{k} p_l(x_l) dx_l \right) \left(\int_{\mathbb{R}^k} g(y_1, \ldots, y_k) \prod_{l=1}^{k} p_l(y_l) dy_l \right)
\end{aligned} \tag{5.90}$$

We go from the first to the second line using the result for $n = 1$ and from the second to the last by the induction hypothesis. □

Notice that if one of the functions f and g in Lemma 5.6 is nondecreasing and the other is nonincreasing, the inequality holds with the opposite sign. We now turn to the proof of Theorem 5.8.

Proof of Theorem 5.8. We work with the rotated hamiltonian with interaction in the 1-3 direction, see Eq. (5.40). We thus prove the result for correlations along the 1st and 3rd direction of spin. In order to prove the theorem, we need to apply Lemma 5.6

to suitable functions, which should have the right monotonicity properties. Firstly, let us consider the partition function $Z^{\mathrm{qu}}_{\Lambda,\beta}$. For any $X \subset \Lambda$ we have:

$$\frac{1}{\beta}\frac{\partial}{\partial J^1_X} Z^{\mathrm{qu}}_{\Lambda,\beta} = \operatorname{Tr} \prod_{x\in X} \mathcal{S}^1_x \, \mathrm{e}^{-\beta H^{\mathrm{qu}}_\Lambda} = Z^{\mathrm{qu}}_{\Lambda,\beta} \left\langle \prod_{x\in X} \mathcal{S}^1_x \right\rangle^{\mathrm{qu}}_{\Lambda,\beta} \geq 0. \tag{5.91}$$

The inequality above holds since we are considering the trace of a matrix with non-negative elements. Secondly, let us focus on the correlation function $\langle \prod_{x\in Y} \mathcal{S}^1_x \rangle^{\mathrm{qu}}_{\Lambda,\beta}$. For any $X, Y \subset \Lambda$ we have, by Corollary 5.3:

$$\frac{\partial}{\partial J^1_X} \left\langle \prod_{x\in Y} \mathcal{S}^1_x \right\rangle^{\mathrm{qu}}_{\Lambda,\beta} \geq 0. \tag{5.92}$$

This shows that both the partition function $Z^{\mathrm{qu}}_{\Lambda,\beta}$ and the correlation function $\langle \prod_{x\in Y} \mathcal{S}^1_x \rangle^{\mathrm{qu}}_{\Lambda,\beta}$ are non decreasing function of the coupling constants J^1_X for any $X \subset \Lambda$. On the other hand, again by Corollary 5.3, for any $X, Y \subset \Lambda$

$$\frac{\partial}{\partial J^1_X} \left\langle \prod_{x\in Y} \mathcal{S}^3_x \right\rangle^{\mathrm{qu}}_{\Lambda,\beta} \leq 0, \tag{5.93}$$

i.e. $\langle \prod_{x\in Y} \mathcal{S}^3_x \rangle^{\mathrm{qu}}_{\Lambda,\beta}$ is a non increasing function of the coupling constant J^1_X for any $X, Y \subset \Lambda$. The following is then a straightforward application of Lemma 5.6, given Eq.s (5.91) and (5.92): for any $X \subset \Lambda$

$$\begin{aligned} \left\langle \prod_{x\in X} \mathcal{S}^1_x \right\rangle^{A}_{\Lambda,\beta} &= \frac{1}{\mathbb{E}_{J^1}\left(Z^{\mathrm{qu}}_{\Lambda,\beta}\right)} \mathbb{E}_{J^1}\left(Z^{\mathrm{qu}}_{\Lambda,\beta} \left\langle \prod_{x\in X} \mathcal{S}^1_x \right\rangle^{\mathrm{qu}}_{\Lambda,\beta} \right) \\ &\geq \mathbb{E}\left(\left\langle \prod_{x\in X} \mathcal{S}^1_x \right\rangle \right) = \left\langle \prod_{x\in X} \mathcal{S}^1_x \right\rangle^{Q}. \end{aligned} \tag{5.94}$$

Analogously from Lemma 5.6 and Eq.s (5.91) and (5.93) we get that, for any $X \subset \Lambda$:

$$\begin{aligned} \left\langle \prod_{x\in X} \mathcal{S}^3_x \right\rangle^{A}_{\Lambda,\beta} &= \frac{1}{\mathbb{E}_{J^1}\left(Z^{\mathrm{qu}}_{\Lambda,\beta}\right)} \mathbb{E}\left(Z_\Lambda \left\langle \prod_{x\in X} \mathcal{S}^3_x \right\rangle^{\mathrm{qu}}_{\Lambda,\beta} \right) \\ &\leq \mathbb{E}_{J^1}\left(\left\langle \prod_{x\in X} \mathcal{S}^3_x \right\rangle^{\mathrm{qu}}_{\Lambda,\beta} \right) = \left\langle \prod_{x\in X} \mathcal{S}^3_x \right\rangle^{Q}_{\Lambda,\beta}. \end{aligned} \tag{5.95}$$

□

Chapter 6

Classical spin systems and random loop representations

Random loop models is the expression generally used to denote models of interacting closed paths on a lattice. There is a great variety of them and they arise in different contexts and in different fields of mathematics and physics. Interestingly, many statistical mechanical models both in the classical and quantum setting are deeply linked to certain loop models, and much effort has been devoted to explore this relationship [14, 77, 4, 81, 63, 65, 78]. In this chapter we focus on O(n) spin systems and their representation in terms of gases of interacting random loops. Firstly, we review the celebrated Brydges-Fröhlich-Spencer representation, a random loop formulation of O(n) models proposed in the seminal work [14]. We then propose an alternative representation which takes inspiration from the random current representation proposed for the Ising model [1], and we show that these two representations are equivalent. The last section is devoted to reformulating some correlation functions for O(n) models in terms of loop properties. The results presented in this chapter are part of a work in progress yet to be published.

6.1 O(n) spin systems

As we have seen in Chapter 2, O(n) models are a class of statistical mechanical models which enjoy a O(n) symmetry. We briefly recall here their definition. Let $(\Lambda, \mathcal{E})$ be the lattice, with $\Lambda \subset\subset \mathbb{Z}^d$ the set of sites and $\mathcal{E}$ the set of edges, and let $n \in \mathbb{N}$, $n \geq 1$. On each site of the lattice let there be a spin, that is a unit vector in $\mathbb{R}^n$: $\vec{\varphi}_x \in \mathbb{S}^{n-1}\ \forall x \in \Lambda$. The spins interact via a rotation invariant hamiltonian: for any spin configuration $\varphi = \{\vec{\varphi}_x\}_{x\in\Lambda}$

$$H_\Lambda^{O(n)}(\varphi) = -\frac{1}{2} \sum_{x,y\in\Lambda} J_{xy}\, \vec{\varphi}_x \cdot \vec{\varphi}_y. \tag{6.1}$$

J_{xy} is a non negative coupling constant for any x, $y \in \Lambda$. In particular, we assume it has the following form, ensuring only nearest neighbours interactions:

$$J_{xy} = J_{yx} \begin{cases} \geq 0 & \text{if } (x,y) \in \mathcal{E}, \\ = 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

The partition function of the model is then defined in the usual way as

$$Z_\Lambda^{O(n)} = \int_{(\mathbb{S}^{n-1})^\Lambda} \prod_{x\in\Lambda} d\vec{\varphi}_x \; \mathrm{e}^{-H_\Lambda^{O(n)}(\varphi)}. \tag{6.3}$$

Here $d\vec{\varphi}$ is the (unormalised) uniform measure over $\mathbb{S}^{n-1}$. Notice the we have assumed $\beta = 1$, and we use this convention for the rest of this chapter, dropping any dependency on β in the notation. The finite volume Gibbs state is defined in the usual way:

$$\langle f \rangle_\Lambda^n = \frac{1}{Z_\Lambda^{O(n)}} \int_{(\mathbb{S}^{n-1})^\Lambda} \prod_{x\in\Lambda} d\vec{\varphi}_x\, f(\varphi)\, \mathrm{e}^{-H_\Lambda^{O(n)}(\phi)}, \tag{6.4}$$

with f any local function over the configuration space. Notice that we are considering free boundary conditions i.e. there is no interaction at the boundary of Λ.

6.2 The Brydges-Fröhlich-Spencer representation of O(n) models

O(n) models are of great importance in statistical mechanics and have been very broadly studied (see [23] for a recent review and references therein). We are particularly interested in the work by Brydges, Fröhlich and Spencer [14], which provides an explicit link between a class of loop models and O(n) spin systems and has been sucessfully exploited in the literature [24, 20]. This map, which takes the name of BFS representation, is reviewed in this section. Let us first introduce some necessary notation for loops.

Definition 6.1 (Loops)**.** *Let $\Lambda \subset\subset \mathbb{Z}^d$. A loop γ of length $\ell \in \mathbb{N}$ is a collection of sites $\gamma = (x_1, x_2, \ldots, x_\ell)$ with $x_i \in \Lambda$ and $(x_i, x_{i+1}) \in \mathcal{E}$ $\forall$ $i \in \{1, \ldots, \ell\}$, with $x_{\ell+1} = x_1$. We identify loops that are the cyclical permutation one of the other –*

$(x_1, \dots, x_{\ell-1}, x_\ell)$ and $(x_\ell, x_1, \dots, x_{\ell-1})$ are the same object. Moreover we identify loops in which the sites are the same but in reversed order, i.e. $(x_1, x_2, \dots, x_\ell)$ and $(x_\ell, x_{\ell-1}, \dots, x_1)$ are the same object. The length of a loop is usually denoted as $\ell(\gamma)$.

We adopt a slightly different notation from the one of the original paper [14]: we choose to use unoriented loops, instead of oriented as in [14]. The set of all such loops in the lattice Λ is denoted by Γ_Λ. We also define open paths:

Definition 6.2 (Open paths)**.** *Let $\Lambda \subset\subset \mathbb{Z}^d$. An open path ω of length $\ell \in \mathbb{N}$ is a collection of sites $\omega = (x_1, x_2, \dots, x_\ell)$ with $x_i \in \Lambda$ for all $i \in \{1, \dots, \ell\}$ and $(x_i, x_{i+1}) \in \mathcal{E}\ \forall\ i \in \{1, \dots, \ell - 1\}$.*

Notation. For any path ω with starting point $x_1 = x$ and ending point $x_\ell = y$ we write $\omega : x \to y$.

It is evident that loops and paths can intersect with themselves and go through the same site of the lattice an arbitrary number of times. This is measured by the so called local time.

Definition 6.3 (Local time)**.** *Let $\eta = (x_1, x_2, \dots, x_\ell)$ be a loop or a path of length $\ell \in \mathbb{N}$ in $\Lambda \subset\subset \mathbb{Z}^d$. The local time $n_x(\eta)$ is the number of times η hits site $x \in \Lambda$: $n_x(\eta) = |\{j \in \{1, \dots, \ell\} : x_j = x\}|$.*

Analogously, the edge local time counts how many times a loop goes through a given edge.

Definition 6.4 (Edge local time)**.** *Let $\gamma = (x_1, x_2, \dots, x_\ell)$ be a loop of length $\ell \in \mathbb{N}$ in $\Lambda \subset\subset \mathbb{Z}^d$. Given $e \in \mathcal{E}$, the edge local time is defined as $t_e(\gamma) = |\{i \in \{1, \dots, \ell\} : (x_i, x_{i+1}) = e\}|$, with the convention that $x_{\ell+1} = x_1$.*

Loops have the additional feature that in principle they can "go in circles" and wind up on themselves – it is then necessary to introduce a winding number for loops.

Definition 6.5 (Winding number)**.** *Let γ be a loop in $(\Lambda, \mathcal{E})$. If it is constituted by a repeated sequence of sites, the winding number $W(\gamma)$ is the number of times the sequence is repeated. The repeated sequence – which is itself a loop with winding number 1 – is called elemental loop, and we denote it by $\hat{\gamma}$.*

Example 6.1. Let γ be a loop of this sort: $\gamma = (x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4)$ with $(x_i, x_{i+1}) \in \mathcal{E}$ for $i = \{1, 2, 3\}$ and $(x_1, x_4) \in \mathcal{E}$; also, $x_1 \neq x_3$ (or $x_2 \neq x_4$). Then $W(\gamma) = 2$. The elemental loop of γ is $\hat{\gamma} = (x_1, x_2, x_3, x_4)$.

Remark. Notice that for any loop $\ell(\gamma) = W(\gamma)\ell(\hat{\gamma})$. Analogously for any loop $\gamma \in \Gamma_\Lambda$ and for any edge $e \in \mathcal{E}$ we have $t_e(\gamma) = W(\gamma)t_e(\hat{\gamma})$.

Remark. Notice that there are many possible ways of formulating the same loop, due to the invariance under cyclicity and reflection. In particular, if a loop has length $\ell(\gamma)$, winding number $W(\gamma)$, and $\ell(\hat{\gamma}) > 2$ (i.e. γ does not live on an edge going back and forth between the two vertices of the edge) there are $2\frac{\ell(\gamma)}{W(\gamma)}$ equivalent ways of writing down the same loop. The factor 2 is due to the fact that we can choose two orientations (clockwise or counterclockwise), while the factor $\frac{\ell(\gamma)}{W(\gamma)}$ counts the number of possible choices of starting point i.e. the number of distinct cyclic permutations given a certain orientation. In case $\ell(\hat{\gamma}) = 2$ the factor 2 disappears because a change in the orientation is equivalent to a cyclic permutation.

Given a loop or a path, we can define explicitly a weight in terms of the coupling constants of the O(n) model defined in Eq. (6.2).

Definition 6.6 (Weight of a path/loop). *Let $\Lambda \subset\subset \mathbb{Z}^d$ with set of edges $\mathcal{E}$, and let η be an open path or a loop of length $\ell \in \mathbb{N}$: $\eta = (x_1, x_2, \ldots, x_\ell)$. Let $\{J_{xy}\}_{(x,y)\in\mathcal{E}}$ be coupling constants for a O(n) model as defined in Eq.* (6.2)*. The weight of η is defined as*

$$\mathcal{J}(\eta) = \prod_{i=1}^{\ell-1} J_{x_i x_{i+1}}$$

if η is an open path and

$$\mathcal{J}(\eta) = \prod_{i=1}^{\ell} J_{x_i x_{i+1}} \quad \textit{with} \quad x_{\ell+1} = x_1$$

if η is a loop.

We can now prove that the partition function of O(n) models can be reformulated as the partition function for a gas of interacting loops.

Theorem 6.1 (Random loop representation of O(n) models [14]). *$Z_\Lambda^{O(n)}$ can be reformulated as follows:*

$$Z_\Lambda^{O(n)} = \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\ldots,\gamma_k \in \Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} e^{-\sum_{x\in\Lambda} \mathcal{V}(n_x)},$$

with $n_x = \sum_{i=1}^{k} n_x(\gamma_i)$ for any loop configuration with k loops. $\mathcal{V}$ is the potential *describing the local interaction amongst loops and has explicit form*

$$e^{-\mathcal{V}(n_x)} = \left(\frac{1}{2}\right)^{n_x} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(n_x + \frac{n}{2}\right)}.$$

Here Γ *is the Gamma function:*

$$\Gamma(q) = \int_0^\infty dt\, t^{q-1} e^{-t}.$$

In order to discuss the proof, we need the following Lemma [14].

Lemma 6.1. *Let* J *be the matrix of coupling constants defined in Eq.* (6.2) *and* L *be a diagonal matrix of the following form:*

$$L_{uv} = \lambda_u \delta_{uv} \ \ \text{with}\ \lambda_u \neq 0 \quad \forall u, v \in \Lambda.$$

Assume $\|L^{-1}J\| < 1$*. Then*

$$\det\left(L - J\right)^{-1} = \left(\prod_{x\in\Lambda} \lambda_x^{-1}\right) \exp\left(\sum_{\gamma\in\Gamma_\Lambda} \frac{\mathcal{J}(\gamma)}{W(\gamma)} 2^{\mathbb{1}(\ell(\hat{\gamma})>2)} \prod_{y\in\Lambda} \lambda_y^{-n_y(\gamma)}\right).$$

Proof of Lemma 6.1. Recall that for any square matrix A we have

$$\det \mathrm{e}^{A} = \mathrm{e}^{\mathrm{Tr}\, A}. \tag{6.5}$$

Then by the properties of the determinant we have

$$\begin{aligned} \det(L-J)^{-1} &= (\det L)^{-1} \left(\det(\mathbb{1} - L^{-1}J)\right)^{-1} \\ &= (\det L)^{-1} \exp\left(-\mathrm{Tr}\ \log(\mathbb{1} - L^{-1}J)\right) \\ &= (\det L)^{-1} \exp\left(\sum_{k\geq 1} \frac{1}{k} \mathrm{Tr}\ \left(L^{-1}J\right)^k\right) \\ &= (\det L)^{-1} \exp\left(\sum_{k\geq 1} \frac{1}{k} \sum_{x_1,\ldots,x_k} \lambda_{x_1}^{-1} J_{x_1x_2} \lambda_{x_2}^{-1} \ldots \lambda_{x_k}^{-1} J_{x_kx_1}\right). \end{aligned} \tag{6.6}$$

Notice that the hypothesis $\|L^{-1}J\| < 1$ is necessary for the right hand side to converge. In the expression above we have used the explicit formulation of L and the power series for the logarithm. By reformulating the expression above through loops we get

$$\det(L-J)^{-1} = (\det L)^{-1} \exp\left(\sum_{\gamma \in \Gamma_\Lambda} \frac{\mathcal{J}(\gamma)}{W(\gamma)} 2^{\mathbb{1}(\ell(\hat{\gamma})>2)} \prod_{x \in \Lambda} \lambda_x^{-n_x(\gamma)} \right). \tag{6.7}$$

The result is thus proved. The expression above follows from the fact that the sites in the sum in the last line of Eq. (6.6) are ordered – thus each loop appears in every possible variant due to cyclicity and orientation. □

Remark. Notice that the condition $\|L^{-1}J\| < 1$ is fulfilled if

$$|\lambda_u| > \sum_{v \in \Lambda} J_{uv} \quad \forall u \in \Lambda. \tag{6.8}$$

Proof of Theorem 6.1. Recall the integral representation of the delta function:

$$\delta(q) = \frac{1}{2\pi} \int_{\mathbb{R}} dt \ \mathrm{e}^{-itq} \quad \forall q \in \mathbb{R}. \tag{6.9}$$

Let us consider the line in the complex plane defined as $C = \{z \in \mathbb{C} : \Im z = -\lambda\}$, with λ chosen big enough so that Eq. (6.8) holds with $\lambda_u = \lambda \ \forall u \in \Lambda$. By a change of variable we have that

$$\delta(q) = \frac{\mathrm{e}^{\lambda}}{2\pi} \int_C dz \, \mathrm{e}^{-izq}. \tag{6.10}$$

It is convenient to reformulate the partition function in the following way:

$$Z_\Lambda^{O(n)} = \int_{(\mathbb{R}^n)^\Lambda} \prod_{x \in \Lambda} d\vec{\varphi}_x \, \delta\left(\vec{\varphi}_x \cdot \vec{\varphi}_x - 1\right) \mathrm{e}^{-H_\Lambda^{O(n)}(\varphi)}. \tag{6.11}$$

By introducing the delta function in the integral above, each spin vector is now integrated over the whole $\mathbb{R}^n$. This will allow us to reformulate the partition function in terms of Gaussian integrals, whose solution is standard. By the integral representation of the delta function in Eq. (6.10) we have

$$Z_\Lambda^{O(n)} = \int_{(\mathbb{R}^n)^\Lambda} \prod_{y \in \Lambda} d\vec{\varphi}_y \prod_{x \in \Lambda} \frac{\mathrm{e}^{\lambda}}{2\pi} \int_C dz_x \ \mathrm{e}^{iz_x} \ \mathrm{e}^{-\frac{1}{2}\sum_{u,v \in \Lambda}(2iz_u\delta_{uv} - J_{uv})\vec{\varphi}_u \cdot \vec{\varphi}_v}. \tag{6.12}$$

We can now proceed with the Gaussian integration in the variables $\{\vec{\varphi}_y\}_{y \in \Lambda}$. Notice that the choice of expressing the δ function as in Eq. (6.10) allows us to use Lemma

6.1 and find

$$
\begin{aligned}
Z_\Lambda^{O(n)} =&(2\pi)^{|\Lambda|\frac{n}{2}} \sum_{k\geq 0} \frac{1}{k!}\left(\frac{n}{2}\right)^k \sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^k \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} 2^{\mathbb{1}(\ell(\hat{\gamma}_i)>2)} \\
&\cdot \prod_{x\in\Lambda} \frac{\mathrm{e}^\lambda}{2\pi}\int_C dz_x \mathrm{e}^{iz_x}(2iz_x)^{-n_z-\frac{n}{2}}.
\end{aligned} \tag{6.13}
$$

Notice that by the definition of the Gamma function

$$
(2iz)^c = \frac{1}{\Gamma(c)}\int_0^\infty dt\, t^{c-1}\mathrm{e}^{-2izt}. \tag{6.14}
$$

This implies that

$$
\begin{aligned}
Z_\Lambda^{O(n)} =& (2\pi)^{|\Lambda|\frac{n}{2}} \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^k \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \\
&\cdot \prod_{x\in\Lambda} \frac{1}{\Gamma\left(n_x+\frac{n}{2}\right)} \frac{\mathrm{e}^\lambda}{2\pi}\int_0^\infty dt_x \int_C dz_x\, \mathrm{e}^{-iz_x(2t_x-1)} t_x^{n_x+\frac{n}{2}-1}.
\end{aligned} \tag{6.15}
$$

Applying once again the integral representation of the delta function from Eq. (6.10) we find

$$
\begin{aligned}
Z_\Lambda^{O(n)} =& (2\pi)^{\frac{n}{2}|\Lambda|} \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^k \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \\
&\cdot \prod_{x\in\Lambda} \left(\frac{1}{2}\right)^{n_x+\frac{n}{2}-1} \frac{1}{\Gamma\left(n_x+\frac{n}{2}\right)} \\
=& (2\pi^{\frac{n}{2}})^{|\Lambda|} \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^k \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \prod_{x\in\Lambda} \left(\frac{1}{2}\right)^{n_x} \frac{1}{\Gamma\left(n_x+\frac{n}{2}\right)}.
\end{aligned} \tag{6.16}
$$

By the definition of the potential $\mathcal{V}$ the result is proved. □

This result shows an exact correspondence between the partition functions of O(n) systems and the ones for a family of gases of interacting random loops. This kind of statement can be generalised also to $2j$-point correlation functions, which are related to a gas of loops and paths together. In order to formulate this statement, we need to introduce some notation.

Notation. Let $j \in \mathbb{N}$. We denote by $\mathscr{P}_{2j}$ the set of all possible pairings of the elements of the set $\{1, 2, \ldots, 2j\}$:

$$\mathscr{P}_{2j} = \Big\{ \{\{k_1, m_1\}, \ldots, \{k_j, m_j\}\} : k_i, m_i \in \{1, \ldots, 2j\} \, \forall i, \\ \{k_l, m_l\} \cap \{k_i, m_i\} = \emptyset \, \forall l \neq i, \, \bigcup_i \{k_i, m_i\} = \{1, \ldots, 2j\} \Big\}, \tag{6.17}$$

with the convention that the order of the $\{k_j, m_j\}$ does not matter. Notice that $|\mathscr{P}_{2j}| = \frac{(2j)!}{2^j j!} = (2j-1)!!$. Given $p \in \mathscr{P}_{2j}$ we denote by $k_i(p), m_i(p)$ the elements constituting the i-th pair of the pairing p.

Example 6.2. The set of possible pairings of four elements is

$$\mathscr{P}_4 = \Big\{ \{\{1,2\}, \{3,4\}\}, \{\{1,3\}, \{2,4\}\}, \{\{1,4\}, \{2,3\}\} \Big\}. \tag{6.18}$$

We can now express the $2j$-point correlation function for the spin $\mathrm{O}(n)$ model as the partition function of a gas of interacting loops and paths.

Theorem 6.2 ($2j$-point correlation function [14])**.** *Let $j \in \mathbb{N}$. Let $x_1, \ldots, x_{2j} \in \Lambda$ and $\alpha_1, \ldots, \alpha_{2j} \in \{1, 2, \ldots, n\}$. Then*

$$Z_\Lambda^{O(n)} \langle \varphi_{x_1}^{\alpha_1} \ldots \varphi_{x_{2j}}^{\alpha_{2j}} \rangle_\Lambda^n = \sum_{p \in \mathscr{P}_{2j}} \sum_{\substack{\omega_1, \ldots, \omega_j: \\ \omega_i : x_{k_i(p)} \to x_{m_i(p)} \\ \forall i \in \{1, \ldots, j\}}} \prod_{i=1}^{j} \mathcal{J}(\omega_i) \delta_{\alpha_{x_{k_i(p)}}, \alpha_{x_{m_i(p)}}} \\ \cdot \sum_{k \geq 0} \frac{n^k}{k!} \sum_{\gamma_1, \ldots, \gamma_k \in \Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} e^{-\sum_{x \in \Lambda} \mathcal{V}(n_x)}$$

with $\mathcal{V}$ the potential describing the local interaction between loops and paths of the form:

$$e^{-\mathcal{V}(n_x)} = \left(\frac{1}{2}\right)^{n_x} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(n_x + \frac{n}{2}\right)}$$

Here for any site x, $n_x = \sum_{i=1}^{j} n_x(\omega_i) + \sum_{l=1}^{k} n_x(\gamma_l)$ for any configuration with k loops and j paths. φ_x^α with $\alpha \in \{1, \ldots, n\}$ is the α-th component of the vector $\vec{\varphi}_x \in \mathbb{R}^n$.

The paths appearing on the right hand side are only those between pairs of sites such that the spins in the $2j$-point correlation functions appear in the same component in those two sites.

Example 6.3. From Theorem 6.2 we have that:

$$Z_\Lambda^{O(n)} \langle \varphi_x^1 \varphi_y^1 \rangle_\Lambda^n = \sum_{\omega\colon x\to y} \mathcal{J}(\omega) \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\dots,\gamma_k \in \Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_{x\in\Lambda} \mathcal{V}(n_x)}. \tag{6.19}$$

Before turning to the proof of Theorem 6.2, we need the following lemma.

Lemma 6.2. *Let L and J be as in Lemma 6.1 with $\|L^{-1}J\| < 1$. Then*

$$\big((L-J)^{-1}\big)_{xy} = \sum_{\omega\colon x\to y} \mathcal{J}(\omega) \prod_{z\in\Lambda} \lambda_z^{-n_z(\omega)}.$$

Proof. Notice that $(L-J)^{-1} = (\mathbb{1} - L^{-1}J)^{-1}L^{-1}$. Moreover, since $\|L^{-1}J\| < 1$ the following power series converges: $(\mathbb{1} - L^{-1}J)^{-1} = \sum_{k\geq 0}(L^{-1}J)^k$. Since L (and hence L^{-1}) is diagonal

$$\big((L-J)^{-1}\big)_{xy} = \sum_{k\geq 0} \Big((L^{-1}J)^k\Big)_{xy} L_{yy}^{-1}. \tag{6.20}$$

If $k = 0$, then $(L^{-1}J)^k = \mathbb{1}$. If $k > 0$ we have that

$$\Big((L^{-1}J)^k\Big)_{xy} = \sum_{u_1,\dots,u_k} L_{xu_1}^{-1} J_{u_1u_2} \dots L_{u_{k-1}u_k}^{-1} J_{u_ky} \tag{6.21}$$

Since L is diagonal we have:

$$\Big((L^{-1}J)^k\Big)_{xy} = \sum_{\substack{u_1,\dots,u_k\in\Lambda \\ u_1=x}} L_{u_1u_1}^{-1} J_{u_1u_2} \dots L_{u_ku_k}^{-1} J_{u_ky}. \tag{6.22}$$

Then

$$\big((L-J)^{-1}\big)_{xy} = \sum_{k\geq 0} \sum_{\substack{u_1,\dots,u_{k+1}: \\ u_1=x,\, u_{k+1}=y}} \prod_{i=1}^{k} J_{u_iu_{i+1}} \prod_{j=1}^{k+1} L_{u_ju_j}^{-1}. \tag{6.23}$$

By the explicit form of J and L we get

$$\begin{aligned} \big((L-J)^{-1}\big)_{xy} &= \sum_{k\geq 0} \sum_{\substack{\omega\colon x\to y \\ \ell(\omega)=k+1}} \mathcal{J}(\omega) \prod_{z\in\Lambda} \lambda_z^{-n_z(\omega)} \\ &= \sum_{\omega\colon x\to y} \mathcal{J}(\omega) \prod_{z\in\Lambda} \lambda_z^{-n_z(\omega)}. \end{aligned} \tag{6.24}$$

□

We are now ready to discuss the proof of Theorem 6.2.

Proof. Recall that

$$Z_\Lambda^{O(n)} \langle \varphi_{x_1}^{\alpha_1} \dots \varphi_{x_{2j}}^{\alpha_{2j}} \rangle_\Lambda^n = \int_{(\mathbb{R}^n)^\Lambda} \prod_{x \in \Lambda} d\vec{\varphi}_x \, \delta(\vec{\varphi}_x \cdot \vec{\varphi}_x - 1) \, \varphi_{x_1}^{\alpha_1} \dots \varphi_{x_{2j}}^{\alpha_{2j}} \, \mathrm{e}^{-H_\Lambda^{O(n)}}. \tag{6.25}$$

As in the previous case we can use the representation of the delta function as an integral over a suitably chosen contour C in the complex plane (see Eq. (6.10)) and find

$$\begin{aligned} Z_\Lambda^{O(n)} \langle \varphi_{x_1}^{\alpha_1} \dots \varphi_{x_{2j}}^{\alpha_{2j}} \rangle_\Lambda^n = & \int_{(\mathbb{R}^n)^\Lambda} \prod_{y \in \Lambda} d\vec{\varphi}_y \\ & \cdot \prod_{x \in \Lambda} \frac{\mathrm{e}^\lambda}{2\pi} \int_C dz_x \, \mathrm{e}^{iz_x} \, \mathrm{e}^{-\frac{1}{2} \sum_{u,v \in \Lambda} M_{uv} \vec{\varphi}_u \cdot \vec{\varphi}_v} \varphi_{x_1}^{\alpha_1} \dots \varphi_{x_{2j}}^{\alpha_{2j}}, \end{aligned} \tag{6.26}$$

where $M_{uv} = (2iz_u \delta_{uv} - J_{uv})$. We can now perform the Gaussian integration and integrate over $\{\vec{\varphi}_x\}_{x \in \Lambda}$. In order to do so we use the following well known result about Gaussian integrals for matrices: let $F(\varphi)$ be some polynomial of various components of the spins $\{\vec{\varphi}_x\}_{x \in \Lambda}$. Then for any $x \in \Lambda$, any $\alpha \in \{1, \dots, n\}$

$$\begin{aligned} & \int_{(\mathbb{R}^n)^\Lambda} \prod_{y \in \Lambda} d\vec{\varphi}_y \, \mathrm{e}^{-\frac{1}{2} \sum_{u,v \in \Lambda} M_{uv} \vec{\varphi}_u \cdot \vec{\varphi}_v} \, \varphi_x^\alpha \, F(\varphi) \\ = & \int_{(\mathbb{R}^n)^\Lambda} \prod_{y \in \Lambda} d\vec{\varphi}_y \, \mathrm{e}^{-\frac{1}{2} \sum_{u,v \in \Lambda} M_{uv} \vec{\varphi}_u \cdot \vec{\varphi}_v} \sum_{w \in \Lambda} \left(M^{-1} \right)_{yw} \frac{\partial}{\partial \varphi_w^\alpha} F(\varphi). \end{aligned} \tag{6.27}$$

By applying this repeatedly to Eq. (6.26) we find

$$\begin{aligned} Z_\Lambda^{O(n)} \langle \varphi_{x_1}^{\alpha_1} \dots \varphi_{x_{2j}}^{\alpha_{2j}} \rangle_\Lambda^n = & \int_{(\mathbb{R}^n)^\Lambda} \prod_{y \in \Lambda} d\vec{\varphi}_y \prod_{x \in \Lambda} \frac{e^\lambda}{2\pi} \int_C dz_x \, \mathrm{e}^{iz_x} \\ & \cdot \, \mathrm{e}^{-\frac{1}{2} \sum_{u,v \in \Lambda} M_{uv} \vec{\varphi}_u \cdot \vec{\varphi}_v} \sum_{p \in \mathscr{P}_{2j}} \prod_{i=1}^{j} \left(M^{-1} \right)_{x_{k_i(p)} x_{m_i(p)}}. \end{aligned} \tag{6.28}$$

By applying Lemma 6.2 in order to have an explicit expression for M^{-1} we find

$$\begin{aligned} Z_\Lambda^{O(n)} \langle \varphi_{x_1}^{\alpha_1} \dots \varphi_{x_{2j}}^{\alpha_{2j}} \rangle_\Lambda^n = & \sum_{p \in \mathscr{P}_{2j}} \prod_{i=1}^{j} \sum_{\omega_i : x_{k_i(p)} \to x_{m_i(p)}} \mathcal{J}(\omega_i) \delta_{\alpha_{k_i(p)}, \alpha_{m_i(p)}} \\ & \cdot \int_{(\mathbb{R}^n)^\Lambda} \prod_{y \in \Lambda} d\vec{\varphi}_y \prod_{x \in \Lambda} \frac{\mathrm{e}^\lambda}{2\pi} \int_C dz_x \, \mathrm{e}^{iz_x} (2iz_x)^{-\sum_{i=1}^n n_x(\omega_i)} \mathrm{e}^{-\frac{1}{2} \sum_{u,v \in \Lambda} M_{uv} \vec{\varphi}_u \cdot \vec{\varphi}_v}. \end{aligned} \tag{6.29}$$

We can now perform the Gaussian integration and by Lemma 6.1 we have

$$Z_\Lambda^{O(n)}\langle\varphi_{x_1}^{\alpha_1}\dots\varphi_{x_{2j}}^{\alpha_{2j}}\rangle_\Lambda^n = (2\pi)^{|\Lambda|\frac{n}{2}}\sum_{p\in\mathscr{P}_{2j}}\prod_{i=1}^{j}\sum_{\omega_i : x_{k_i(p)}\to x_{m_i(p)}}\mathcal{J}(\omega_i)\delta_{\alpha_{k_i(p)},\alpha_{m_i(p)}}$$
$$\cdot\sum_{k\geq 0}\frac{1}{k!}\left(\frac{n}{2}\right)^k\sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda}\prod_{i=1}^{k}\frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)}2^{\mathbb{1}(\ell(\hat{\gamma}_i)>2)}\prod_{x\in\Lambda}\frac{\mathrm{e}^\lambda}{2\pi}\int_C dz_x\mathrm{e}^{iz_x}(2iz_x)^{-n_x-\frac{n}{2}}. \tag{6.30}$$

Here we have introduced the notation $n_x = \sum_{i=1}^k n_x(\gamma_i) + \sum_{i=1}^j n_x(\omega_i)$. The result now follows rewriting the powers of $\{z_x\}_{x\in\Lambda}$ via

$$(2iz)^c = \frac{1}{\Gamma(c)}\int_0^\infty dz\, z^{c-1}\mathrm{e}^{-2izt} \tag{6.31}$$

and then integrating over $\{z_x\}_{x\in\Lambda}$ as in the proof of Theorem 6.1. □

6.2.1 BFS as a measure over loop configurations

So far we have examined correlation functions and partition functions for O(n) models as partition functions of gases of random loops and paths. We now aim to define a measure over loops with $Z_\Lambda^{O(n)}$ as normalisation. Recall that by Theorem 6.1

$$Z_\Lambda^{O(n)} = \sum_{k\geq 0}\frac{n^k}{k!}\sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda}\prod_{i=1}^{k}\frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)}\mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x)} \tag{6.32}$$

with

$$\mathrm{e}^{-\mathcal{V}(n_x)} = \left(\frac{1}{2}\right)^{n_x}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(n_x+\frac{n}{2}\right)}. \tag{6.33}$$

In the expression above, each configuration can have an arbitrarily high number of loops (i.e. we sum over all possible values of $k\in\mathbb{N}$, which counts how many loops there are in a configuration), and the same loop can appear more than once in the same configuration. In order to make this more precise, we need to define a configuration space for loops as follows.

Definition 6.7 (Loop configurations and occupation numbers). *The set of all possible loop configurations on* $(\Lambda,\mathcal{E})$ *is*

$$\mathcal{R}_\Lambda = \left\{\{r_\gamma\}_{\gamma\in\Gamma_\Lambda} :\ r_\gamma\in\mathbb{N}\ \forall\gamma\in\Gamma_\Lambda,\ \sum_{\gamma\in\Gamma_\Lambda} r_\gamma<\infty\right\}.$$

We introduce the notation $\{r_\gamma\}_{\gamma\in\Gamma_\Lambda} = \mathbf{r}$. *Given* $\mathbf{r} \in \mathcal{R}_\Lambda$, r_γ *is called occupation number of* γ.

The idea of the definition above is to have unordered configurations of loops, where each $\gamma \in \Gamma_\Lambda$ appears r_γ times. The constraint about $\sum_{\gamma\in\Gamma_\Lambda} r_\gamma$ is necessary since each loop configuration has a finite, though arbitrarily high, number of loops. It is then easy to check that the following is a well defined and normalised measure over $\mathcal{R}_\Lambda$:

$$\mu_\Lambda^n(\mathbf{r}) = \frac{1}{Z_\Lambda^{O(n)}} \prod_{\gamma\in\Gamma_\Lambda} \frac{n^{r_\gamma}}{r_\gamma!} \left(\frac{\mathcal{J}(\gamma)}{W(\gamma)}\right)^{r_\gamma} \left(\frac{1}{2}\right)^{r_\gamma \mathbb{1}(\ell(\hat{\gamma})=2)} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x(\mathbf{r}))}, \tag{6.34}$$

with $n_x(\mathbf{r}) = \sum_{\gamma\in\Gamma_\Lambda} r_\gamma n_x(\gamma)$ and $\mathcal{V}$ as in Eq. (6.33). Notice that in this setting, in a configuration $\mathbf{r} \in \mathcal{R}_\Lambda$ the r_γ copies of a loop $\gamma \in \Gamma_\Lambda$ are indistinguishable. In the next section we propose a different approach for these loop models where loops and loop configurations are redefined in such a way that there is no ambiguity when the same loop appears more than once. In order to do so, the main focus of this approach is not on the loops themselves, but on the number of times each edge is occupied by a loop. The idea is inspired by the celebrated *random current representation* of the Ising model, introduced by Aizenman [1] and used extensively in the literature [74, 2, 3]. The next section is therefore devoted to a generalisation of this very well studied model. See Appendix C for a brief review of the original random current representation.

6.3 A generalised random current representation for O(n) models

Let us now introduce some necessary notation. Intuitively, the idea is that on each edge of the lattice there is a certain number of *links*.

Definition 6.8 (Link configurations)**.** *Let* $\Lambda \subset\subset \mathbb{Z}^d$ *be a set of sites with edges* $\mathcal{E}$. *A link configuration is a collection of non negative integers* $\bar{m} = \{m_e\}_{e\in\mathcal{E}}$. *Equivalently,* $\bar{m} \in \mathbb{N}^{\mathcal{E}}$.

Given $\bar{m} \in \mathbb{N}^{\mathcal{E}}$, for any $e \in \mathcal{E}$ we say that there are m_e links on it.

Remark. We view a link configuration as a collection of labelled objects attached to edges. That is, edge $e \in \mathcal{E}$ contains links 1, 2, $\dots$, m_e. Each specific link is

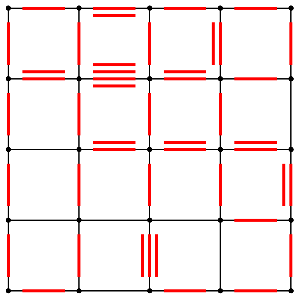

Figure 6.1: An example of $\bar{m}$ such that $\partial\bar{m} = \emptyset$.

then denoted with the notation (e, n_e), with $n_e \leq m_e$. Labels are necessary for the pairings introduced below (Def.s 6.11, 6.12).

Definition 6.9 (Local current)**.** *For any $x \in \Lambda$ and $\bar{m} \in \mathbb{N}^{\mathcal{E}}$, the local current $\mathcal{N}_x$ is the number of links with one ending point in x: $\mathcal{N}_x = \sum_{e \in \mathcal{E}:\, e \ni x} m_e$.*

Definition 6.10 (Sources)**.** *Given $\bar{m} \in \mathbb{N}^{\mathcal{E}}$, $x \in \Lambda$ is a source for $\bar{m}$ if $\mathcal{N}_x$ is odd. The set of all sources of $\bar{m}$ is denoted by $\partial\bar{m} = \{x \in \Lambda : \mathcal{N}_x \text{ is odd.}\}$.*

See Fig. 6.1 for an example of link configuration with $\partial\bar{m} = \emptyset$. Notice that configurations with $\partial\bar{m} = \emptyset$ have one very important feature in common with loop configurations described previously: each site is "touched" by an even number of links, which is precisely what happens with loops. Indeed, since loops are closed, every time they go towards a site, they also have to leave it. Of course, link configurations as defined so far are *not* equivalent to loop configurations because no trajectory has been specified for loops. In order to make the two pictures more similar, one has to specify how to "join" links which have an ending point in common.

Definition 6.11 (Pairings)**.** *Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ with $\partial\bar{m} = \emptyset$. A pairing π for $\bar{m}$ is a choice of pairing of the $\mathcal{N}_x$ links touching x for each $x \in \Lambda$. Notice that at each site x there are $(\mathcal{N}_x - 1)!!$ possible choices. The set of all pairings for $\bar{m}$ is denoted by $\mathcal{P}_{\bar{m}}$.*

Figure 6.2 helps clarifying this definition. See Figure 6.3 for an example of pairing for the link configuration depicted in Figure 6.1.

Definition 6.12 (Pairing function)**.** *Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ with $\partial\bar{m} = \emptyset$ and $\pi \in \mathcal{P}_{\bar{m}}$. Let (e_1, n_{e_1}), (e_2, n_{e_2}) be two links with $e_1 \cap e_2 \neq \emptyset$ and $(e_1, n_{e_1}) \neq (e_2, n_{e_2})$ – i.e. they*

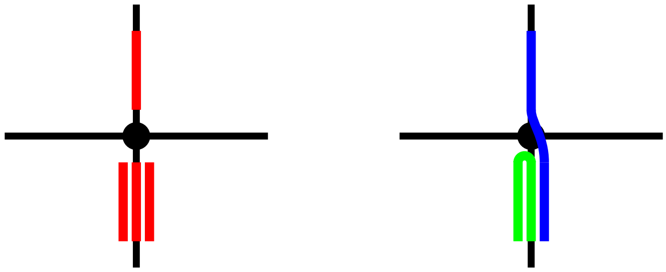

Figure 6.2: A link configuration around a site and a possible choice of pairing for it. Notice that the number of links around the site is even.

are two distinct links with (at least) one site in common. The pairing function $P_\pi\left((e_1, n_{e_1}), (e_2, n_{e_2})\right)$ *is defined as:*

$$P_\pi\left((e_1, n_{e_1}), (e_2, n_{e_2})\right) = \begin{cases} 1 & \textit{if } (e_1, n_{e_1}) \textit{ and } (e_2, n_{e_2}) \textit{ are paired by } \pi; \\ 0 & \textit{otherwise.} \end{cases}$$

We can now define current configurations.

Definition 6.13 (Current configurations)**.** *A current configuration on* $(\Lambda, \mathcal{E})$ *is a choice of a link configuration and a pairing* $(\bar{m}, \pi)$ *with* $\bar{m} \in \mathbb{N}^{\mathcal{E}}$*,* $\partial\bar{m} = \emptyset$ *and* $\pi \in \mathcal{P}_{\bar{m}}$*. The set of all possible configurations of this type is denoted by* $\mathcal{C}_\Lambda$*:*

$$\mathcal{C}_\Lambda = \{(\bar{m}, \pi) : \bar{m} \in \mathbb{N}^{\mathcal{E}}, \partial\bar{m} = \emptyset, \pi \in \mathcal{P}_{\bar{m}}\}.$$

It is clear that, given $(m, \pi) \in \mathcal{C}_\Lambda$, loops emerge naturally. See Figure 6.3 for an example. These loops can be formally defined as follows.

Definition 6.14. *Let* $(\bar{m}, \pi) \in \mathcal{C}_\Lambda$*. The set of loops* $\tau(\bar{m}, \pi) \in \mathcal{R}_\Lambda$ *is defined as* $\tau(\bar{m}, \pi) = \mathbf{r}$ *such that for each* $\gamma \in \Gamma_\Lambda$*,* $\gamma = (x_1, x_2, \ldots, x_\ell)$ *there are* r_γ *distinct sequences* $((e_1, n_{e_1}), \ldots, (e_\ell, n_{e_\ell}))$ *of* ℓ *links of* $(\bar{m}, \pi)$ *with the following properties:*

1. $((e_1, n_{e_1}), \ldots, (e_\ell, n_{e_\ell}))$ *is such that, for any* $i \in \{1, \ldots, \ell\}$*,* $e_i = (x_i, x_{i+1})$ *with* $x_{\ell+1} = x_1$*.*

2. *For any* $i \in \{1, \ldots, \ell\}$*,* $e_i \cap e_{i+i} \neq \emptyset$*,* $(e_i, n_{e_i}) \neq (e_{i+1}, n_{e_{i+1}})$ *and* $P_\pi((e_i, n_{e_i}),$ $(e_{i+1}, n_{e_{i+1}})) = 1$ *with* $(e_{\ell+1}, n_{e_{\ell+1}}) = (e_1, n_{e_1})$*.*

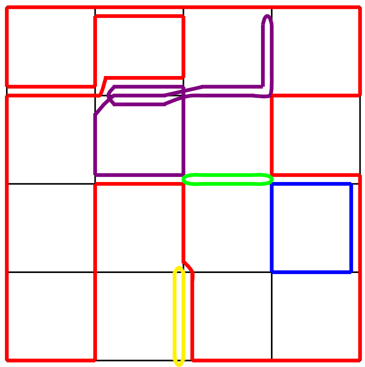

Figure 6.3: A possible pairing for $\bar{m}$ of Figure 6.1. Different loops are depicted in different colours.

Figure 6.4: (a): example of configuration $\bar{m}$ with $\partial \bar{m} = \emptyset$. (b),(c),(d): pairings for $\bar{m}$ reproduced in (a) which belong to $\tau^{-1}(\{\mathbf{r}\})$ for the same $\mathbf{r} \in \mathcal{R}_\Lambda$.

Notice that, due to the constraint $\partial \bar{m} = \emptyset$ and by the definition of pairing, each link appears exactly in one loop only. The map defined above $\tau : \mathcal{C}_\Lambda \to \mathcal{R}_\Lambda$ is onto but not one-to-one. Moreover, given $\mathbf{r} = \tau(\bar{m}, \pi)$, $m_e = \sum_{\gamma \in \Gamma_\Lambda} t_e(\gamma) r_\gamma$, where $t_e(\gamma)$ is the local edge time defined in Def. 6.4. It is then evident that $\tau^{-1}(\{\mathbf{r}\})$ is given by a set of current configurations which have the same link configuration $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ but different pairings. See Fig. 6.4 to clarify this. We denote by $\mathcal{L}(\bar{m}, \pi)$ the number of loops in $(\bar{m}, \pi)$, i.e. given $\mathbf{r} = \tau(\bar{m}, \pi)$, $\mathcal{L}(\bar{m}, \pi) = \sum_{\gamma \in \Gamma_\Lambda} r_\gamma$. We now define a suitable measure over $\mathcal{C}_\Lambda$.

Definition 6.15. *Let $(\Lambda, \mathcal{E})$ be a finite lattice. Let $\{J_{xy}\}_{(x,y)\in\mathcal{E}}$ be non negative coupling constants as defined in Eq.* (6.2)*. The following measure over $\mathcal{C}_\Lambda$ is defined:*

$$\nu^n_\Lambda(\bar{m}, \pi) = \frac{1}{\mathcal{Z}^n_\Lambda} n^{\mathcal{L}(\bar{m},\pi)} \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!} e^{-\sum_{x\in\Lambda} \mathcal{V}(\frac{\mathcal{N}_x}{2})},$$

where $\mathcal{V}$ is the following local potential:

$$e^{-\mathcal{V}\left(\frac{\mathcal{N}_x}{2}\right)} = \left(\frac{1}{2}\right)^{\frac{\mathcal{N}_x}{2}} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{\mathcal{N}_x}{2}+\frac{n}{2}\right)}.$$

$\mathcal{Z}_\Lambda^n$ is the normalisation

$$\mathcal{Z}_\Lambda^n = \sum_{(\bar{m},\pi)\in\mathcal{C}_\Lambda} n^{\mathcal{L}(\bar{m},\pi)} \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!} e^{-\sum_{x\in\Lambda}\mathcal{V}\left(\frac{\mathcal{N}_x}{2}\right)}.$$

In the expressions above, n is a positive parameter.

We are now ready to explore the relationship between μ_Λ^n and ν_Λ^n.

Theorem 6.3. *Let $Z_\Lambda^{O(n)}$ and $\mathcal{Z}_\Lambda^n$ be the two normalisations of the measures μ_Λ^n and ν_Λ^n respectively. Then $Z_\Lambda^{O(n)} = \mathcal{Z}_\Lambda^n$.*

This statement implies that our generalised setting describing links and pairings provides indeed a *generalised random current representation* for O(n) models.

Remark. In the case $n = 1$, $\mathcal{Z}_\Lambda^n$ reduces precisely to the partition function of the Ising model formulated via the random current representation reviewed in Appendix C – see Theorem C.1. Our generalised random current representation is thus a novel result for $n \geq 2$.

The next theorem establishes the equivalence between ν_Λ^n and μ_Λ^n in a stronger sense.

Theorem 6.4. *Let $\tau : \mathcal{C}_\Lambda \to \mathcal{R}_\Lambda$ be the map described above. Then for any $\mathbf{r} \in \mathcal{R}_\Lambda$*

$$\mu_\Lambda^n(\mathbf{r}) = \nu_\Lambda^n(\tau^{-1}(\{\mathbf{r}\})).$$

The next section is devoted to some necessary lemmas and to the proofs of these statements. Before getting to the proofs, a remark on the meaning of this theorem. According to this statement, the map $\tau : \mathcal{C}_\Lambda \to \mathcal{R}_\Lambda$ preserves the volumes. Equivalently, for any $\mathbf{r} \in \mathcal{R}_\Lambda$ the probability of $\mathbf{r}$ according to μ_Λ^n is equal to the probability of $\tau^{-1}(\{\mathbf{r}\}) \subset \mathcal{C}_\Lambda$ according to ν_Λ^n. The existence of a map τ with these properties ensures that these two loop representations are in some sense equivalent.

6.3.1 Proofs of Theorems 6.3 and 6.4

This section collects the proofs of Theorems 6.3, 6.4 and of some necessary preliminary results.

The main ingredient necessary for the proofs of Theorems 6.3 and 6.4 is an explicit estimate of the size of the preimage of the map τ. This is provided by the following statement.

Theorem 6.5. *Let $\tau : \mathcal{C}_\Lambda \to \mathcal{R}_\Lambda$ as described above. Then for any $\mathbf{r} \in \mathcal{R}_\Lambda$*

$$|\tau^{-1}(\{\mathbf{r}\})| = \prod_{e\in\mathcal{E}} t_e(\mathbf{r})! \prod_{\gamma\in\Gamma_\Lambda} \frac{1}{r_\gamma! W(\gamma)^{r_\gamma}} \left(\frac{1}{2}\right)^{r_\gamma \mathbb{1}(\ell(\hat{\gamma})=2)},$$

with $t_e(\mathbf{r}) = \sum_{\gamma\in\Gamma_\Lambda} r_\gamma t_e(\gamma)$.

Recall that $t_e(\gamma)$ is the local edge time, as defined in Def. 6.4. In order to prove this statement, we need to control what happens in the simpler case when we are dealing with configurations with one loop only.

Lemma 6.3. *Let $\mathcal{R}^1_\Lambda = \{\mathbf{r} \in \mathcal{R}_\Lambda : \sum_{\gamma\in\Gamma_\Lambda} r_\gamma = 1\}$, i.e. $\mathcal{R}^1_\Lambda \subset \mathcal{R}_\Lambda$ is the set of loop configurations with one loop only. Let $\boldsymbol{\gamma} \in \mathcal{R}^1_\Lambda$ denote the loop configuration with only a given loop $\gamma \in \Gamma_\Lambda$. Then the following statements hold.*

(a) If $W(\gamma) = 1$ and $\ell(\gamma) > 2$

$$\tau^{-1}(\{\boldsymbol{\gamma}\}) = \prod_{e\in\mathcal{E}} t_e(\gamma)!.$$

(b) If $W(\gamma) > 1$ and $\ell(\hat{\gamma}) > 2$

$$\tau^{-1}(\{\boldsymbol{\gamma}\}) = \frac{1}{W(\gamma)} \prod_{e\in\mathcal{E}} t_e(\gamma)!.$$

(c) If $\ell(\hat{\gamma}) = 2$

$$\tau^{-1}(\{\boldsymbol{\gamma}\}) = (\ell(\gamma) - 1)!.$$

Remark. Notice that the three statements above can be collected together in a unique expression i.e. for any $\boldsymbol{\gamma} \in \mathcal{R}^1_\Lambda$ with only the loop γ present,

$$\tau^{-1}(\{\boldsymbol{\gamma}\}) = \frac{\prod_{e\in\mathcal{E}} t_e(\gamma)!}{W(\gamma)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma})=2)}. \tag{6.35}$$

Proof of Lemma 6.3, Statement (a). Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ be such that $t_e(\gamma) = m_e$ for any $e \in \mathcal{E}$. Let us fix arbitrarily a starting point and an orientation (clockwise or counterclockwise) for the loop $\gamma = (x_1, \ldots, x_\ell)$. Each pairing $\pi \in \mathcal{P}_{\bar{m}}$ such that $(\bar{m}, \pi) \in \tau^{-1}(\{\boldsymbol{\gamma}\})$ is equivalent to a different *peeling* of the loop from the link configuration $\bar{m}$, where a peeling is defined as the following procedure. Let us consider the starting point x_1 and follow the trajectory of γ by jumping from x_i to x_{i+1} (with the convention $x_{\ell+1} = x_1$) choosing one of the links in the edge (x_i, x_{i+1}), with the constraint that we can not choose the same link twice. This provides us with a sequence of links $((e_1, n_{e_1}), \ldots, (e_\ell, n_{e_\ell}))$ with $e_i = (x_i, x_{i+1})$ and $(e_i, n_{e_i}) \neq (e_j, n_{e_j})$ if $i \neq j$. This is equivalent to the pairing π such that $P_\pi\left((e_i, n_{e_i}), (e_{i+1}, n_{e_{i+1}})\right) = 1$ for any $i \in \{1, \ldots, \ell\}$. We now need to count how many such peelings there are. Let us fix an edge e. The first time the peeling hits it, there are m_e possible ways of choosing a link. The second time there are $m_e - 1$, since one link has already been chosen. This procedure can be iterated and the last time the edge is hit by the peeling there is only one link left. By applying this reasoning to every edge, it is clear that there are $\prod_e m_e!$ such peelings, hence the result. □

Proof of Lemma 6.3, Statement (b). Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ be such that $t_e(\gamma) = m_e$ for any $e \in \mathcal{E}$. Let us fix arbitrarily a starting point and an orientation (clockwise or counterclockwise) for the loop $\gamma = (x_1, \ldots, x_\ell)$. Let us denote by $\hat{\ell}$ the length of the elemental loop $\hat{\gamma}$ i.e. $\ell = W(\gamma)\hat{\ell}$. Then $\gamma = (x_1, \ldots, x_{\hat{\ell}}, x_1 \ldots, x_{\hat{\ell}}, \ldots)$. Moreover, notice that $m_e = W(\gamma) t_e(\hat{\gamma})$ for each $e \in \mathcal{E}$. A peeling of such a loop is equivalent to $W(\gamma)$ successive peelings of the elemental loop $\hat{\gamma}$ with the constraint that the links chosen in previous peelings can't be chosen in the following ones. Doing so we get an ordered sequence of $W(\gamma)$ sequences of $\hat{\ell}$ links: $\Big((e_1, n_{e_1,1}), \ldots, (e_{\hat{\ell}}, n_{e_{\hat{\ell}},1})\Big), \ldots, \Big((e_1, n_{e_1,W(\gamma)}), \ldots, (e_{\hat{\ell}}, n_{e_{\hat{\ell}},W(\gamma)})\Big)$. On each edge, each peeling picks $t_e(\hat{\gamma})$ links. These can then appear in different order inside the peeling, as described in the proof of Statement (a). This means that there are

$$\prod_{e \in \mathcal{E}} \binom{m_e}{\underbrace{t_e(\hat{\gamma})\ t_e(\hat{\gamma})\ \cdots\ t_e(\hat{\gamma})}_{W(\gamma) \text{ times}}} \overbrace{t_e(\hat{\gamma})! t_e(\hat{\gamma})! \cdots t_e(\hat{\gamma})!}^{W(\gamma) \text{ times}} = \prod_e m_e! \tag{6.36}$$

different such peelings of the loop γ. The first factor counts the number of possible ways to attribute links to each peeling of $\hat{\gamma}$, while the second counts the number of possible ways in which the chosen links appear in a given peeling of $\hat{\gamma}$. Each of these peelings corresponds to the pairing $\pi \in \mathcal{P}_{\bar{m}}$ with $(\bar{m}, \pi) \in \tau^{-1}(\{\boldsymbol{\gamma}\})$ such that

$$\begin{aligned} P_\pi\left((e_i, n_{e_i,j}), (e_{i+1}, n_{e_{i+1},j})\right) &= 1 \text{ for any } i \in \{1, \ldots, \hat{\ell} - 1\} \\ &\quad \text{and } j \in \{1, \ldots, W(\gamma)\}, \qquad (6.37) \\ P_\pi\Big((e_{\hat{\ell}}, n_{e_{\hat{\ell}},j}), (e_1, n_{e_1,j+1})\Big) &= 1 \text{ for any } j \in \{1, \ldots, W(\gamma)\} \\ &\quad \text{with } n_{e_1,W(\gamma)+1} = n_{e_1,1}. \qquad (6.38) \end{aligned}$$

Notice that the same pairing π corresponds to $W(\gamma)$ peelings: for a given sequence of $W(\gamma)$ peelings of $\hat{\gamma}$, $\Big((e_1, n_{e_1,1}), \ldots, (e_{\hat{\ell}}, n_{e_{\hat{\ell}},1})\Big)$, ..., $\Big((e_1, n_{e_1,W(\gamma)}), \ldots,$ $(e_{\hat{\ell}}, n_{e_{\hat{\ell}},W(\gamma)})\Big)$, any cyclical permutation of the order of the peelings is equivalent to the same pairing, due to the condition in Eq. (6.38). There are $W(\gamma)$ such permutations. This implies Statement (b). □

Proof of Lemma 6.3, Statement (c). Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ such that $m_e = t_e(\gamma)$ for any $e \in \mathcal{E}$. Notice that $\ell(\hat{\gamma}) = 2$ means that γ lives on some edge $e^* = (x_1, x_2)$ and is of the form $\gamma = (x_1, x_2, \ldots, x_1, x_2)$, with the pair (x_1, x_2) repeated $W(\gamma)$ times. Hence $m_e = 0$ for any $e \neq e^*$ and $m_{e^*} = \ell(\gamma) = 2W(\gamma)$. In this scenario, the number of possible pairings $\pi \in \mathcal{P}_{\bar{m}}$ such that $(\bar{m}, \pi) \in \tau^{-1}(\{\boldsymbol{\gamma}\})$ is the number of ways of ordering the $\ell(\gamma)$ links, identifying all the permutations which are cyclical transformations one of the other. There are $(\ell(\gamma) - 1)!$ such permutations. The statement is thus proved.

□

We now have all we need to prove Theorem 6.5.

Proof of Theorem 6.5. Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ such that $m_e = t_e(\mathbf{r})$ for all $e \in \mathcal{E}$. We need to evaluate the number of pairings $\pi \in \mathcal{P}_{\bar{m}}$ such that $(\bar{m}, \pi) \in \tau^{-1}(\{\mathbf{r}\})$. For any $\gamma \in \Gamma_\Lambda$ and any $e \in \mathcal{E}$, each of the r_γ copies goes through e a number of times equal to $t_e(\gamma)$. So to each copy of each loop γ we can associate $t_e(\gamma)$ links on the edge e. In the configuration $\mathbf{r}$ there are $\sum_{\gamma \in \Gamma_\Lambda} r_\gamma$ loops. If we consider them all to be distinguishable, the number of possible ways of associating links to the loops that go through them is

$$\prod_{e \in \mathcal{E}} \frac{m_e!}{\prod_{\gamma \in \Gamma_\Lambda} (t_e(\gamma)!)^{r_\gamma}}. \tag{6.39}$$

This is not completely correct, though. While different loops are indeed distinguishable, copies of the same loop are not, so the total number of possible ways of distributing the links among all the loops in $\mathbf{r}$ is

$$\left(\prod_{\gamma\in\Gamma_\Lambda}\frac{1}{r_\gamma!}\right)\left(\prod_{e\in\mathcal{E}}\frac{m_e!}{\prod_{\gamma\in\Gamma_\Lambda}(t_e(\gamma)!)^{r_\gamma}}\right). \tag{6.40}$$

Once that links have been attributed to loops, we can pair them by a peeling procedure in order to find all the pairings $\pi\in\mathcal{P}_{\bar m}$ with $(\bar m,\pi)\in\tau^{-1}(\{\mathbf{r}\})$. For each copy of each loop, this is equivalent to examining the situation with one loop only as described in Lemma 6.3. So we have

$$\tau^{-1}(\{\mathbf{r}\})=\left(\prod_{e\in\mathcal{E}}\frac{m_e!}{\prod_{\gamma\in\Gamma_\Lambda}(t_e(\gamma)!)^{r_\gamma}}\right)\left(\prod_{\gamma\in\Gamma_\Lambda}\frac{|\tau^{-1}(\{\boldsymbol{\gamma}\})|^{r_\gamma}}{r_\gamma!}\right), \tag{6.41}$$

where we have denoted by $\boldsymbol{\gamma}$ the loop configuration in $\mathcal{R}^1_\Lambda$ with only one copy of the loop γ. The statement follows from Lemma 6.3. □

We are now ready to discuss the proofs of Theorems 6.3 and 6.4.

Proof of Theorem 6.3. Notice that $Z^{O(n)}_\Lambda$ can be formulated in terms of loop configurations $\mathbf{r}\in\mathcal{R}_\Lambda$:

$$Z^{O(n)}_\Lambda=\sum_{\mathbf{r}\in\mathcal{R}_\Lambda}\prod_{\gamma\in\Gamma_\Lambda}\frac{n^{r_\gamma}}{r_\gamma!}\left(\frac{\mathcal{J}(\gamma)}{W(\gamma)}\right)^{r_\gamma}\left(\frac{1}{2}\right)^{r_\gamma\mathbb{1}(\ell(\hat\gamma)=2)}\mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x(\mathbf{r}))} \tag{6.42}$$

Recall that for any $\mathbf{r}\in\mathcal{R}_\Lambda$, all the current configurations such that $(\bar m,\pi)\in\tau^{-1}(\{\mathbf{r}\})$ share the same link configuration $\bar m$. Moreover notice that for any $(\bar m,\pi)\in\tau^{-1}(\{\mathbf{r}\})$ we have that $n_x(\mathbf{r})=\frac{\mathcal{N}_x}{2}$ and $\prod_e J_e^{m_e}=\prod_\gamma\mathcal{J}(\gamma)^{r_\gamma}$. Therefore we have

$$Z^{O(n)}_\Lambda=\sum_{\substack{\bar m\in\mathbb{N}^{\mathcal{E}}:\\ \partial\bar m=\emptyset}}\prod_e J_e^{m_e}\sum_{\substack{\mathbf{r}\in\mathcal{R}_\Lambda:\,\forall e\in\mathcal{E}\\ t_e(\mathbf{r})=m_e}}\prod_{\gamma\in\Gamma_\Lambda}\frac{n^{r_\gamma}}{r_\gamma!W(\gamma)^{r_\gamma}}\left(\frac{1}{2}\right)^{-r_\gamma\mathbb{1}(\ell(\hat\gamma)=2)}\mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(\frac{\mathcal{N}_x}{2})}. \tag{6.43}$$

We can now introduce pairings and find

$$\begin{aligned}Z^{O(n)}_\Lambda=&\sum_{\substack{\bar m\in\mathbb{N}^{\mathcal{E}}:\\ \partial\bar m=\emptyset}}\prod_e J_e^{m_e}\sum_{\substack{\mathbf{r}\in\mathcal{R}_\Lambda:\,\forall e\in\mathcal{E}\\ t_e(\mathbf{r})=m_e}}\sum_{\substack{\pi\in\mathcal{P}_{\bar m}:\\ (\bar m,\pi)\in\tau^{-1}(\mathbf{r})}}\frac{1}{|\tau^{-1}(\{\mathbf{r}\})|}\\ &\cdot\prod_{\gamma\in\Gamma_\Lambda}\frac{n^{r_\gamma}}{r_\gamma!W(\gamma)^{r_\gamma}}\left(\frac{1}{2}\right)^{r_\gamma\mathbb{1}(\ell(\hat\gamma)=2)}\mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(\frac{\mathcal{N}_x}{2})}.\end{aligned} \tag{6.44}$$

Given the definition of τ this implies

$$
\begin{aligned}
Z_\Lambda^{O(n)} = \sum_{(\bar{m},\pi)\in\mathcal{C}_\Lambda} \frac{\mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}\left(\frac{\mathcal{N}_x}{2}\right)}\, n^{\mathcal{L}(\bar{m},\pi)}}{|\tau^{-1}(\{\tau(\bar{m},\pi)\})|} \prod_{e\in\mathcal{E}} J_e^{m_e} \\
\cdot \prod_{\gamma\in\Gamma_\Lambda} \left(\frac{1}{2}\right)^{-r_\gamma^{(\bar{m},\pi)}\mathbb{1}(\ell(\hat{\gamma})=2)} \frac{1}{W(\gamma)^{r_\gamma^{(\bar{m},\pi)}}\, r_\gamma^{(\bar{m},\pi)}!},
\end{aligned}
\tag{6.45}
$$

with $\mathbf{r}^{(\bar{m},\pi)} = \tau(\bar{m},\pi)$. The result follows from Theorem 6.5. ☐

Proof of Theorem 6.4. Let $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ such that $m_e = t_e(\mathbf{r})$. Then for any $\mathbf{r} \in \mathcal{R}_\Lambda$

$$
\nu_\Lambda^n(\{\tau^{-1}(\{\mathbf{r}\})\}) = \frac{1}{\mathcal{Z}_\Lambda^n} \sum_{\substack{\pi\in\mathcal{P}_{\bar{m}}:\\ (\bar{m},\pi)\in\tau^{-1}(\{\mathbf{r}\})}} n^{\mathcal{L}(\bar{m},\pi)} \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}\left(\frac{\mathcal{N}_x}{2}\right)}. \tag{6.46}
$$

Notice that $\mathcal{L}(\bar{m},\pi)$ is the same for all π such that $(\bar{m},\pi) \in \tau^{-1}(\{\mathbf{r}\})$ and is equal to $\sum_{\gamma\in\Gamma_\Lambda} r_\gamma$. By the definition of τ, $|\{\pi \in \mathcal{P}_{\bar{m}} : (\bar{m},\pi) \in \tau^{-1}(\{\mathbf{r}\})\}| = |\tau^{-1}(\{\mathbf{r}\})|$. Moreover, recall that for any $(\bar{m},\pi) \in \tau^{-1}(\{\mathbf{r}\})$, $\frac{\mathcal{N}_x}{2} = n_x(\mathbf{r})$ for any site x and $\prod_e J_e^{m_e} = \prod_\gamma \mathcal{J}(\gamma)^{r_\gamma}$. Then

$$
\nu_\Lambda^n(\{\tau^{-1}(\{\mathbf{r}\})\}) = \frac{1}{\mathcal{Z}_\Lambda^n} |\tau^{-1}(\{\mathbf{r}\})| \prod_{\gamma\in\Gamma_\Lambda} n^{r_\gamma} \mathcal{J}(\gamma)^{r_\gamma} \prod_{e\in\mathcal{E}} \frac{1}{t_e(\mathbf{r})!} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x(\mathbf{r}))}. \tag{6.47}
$$

By Theorems 6.3 and 6.5, and by the explicit expression of μ_Λ^n in Eq. (6.34)

$$
\nu_\Lambda^n(\{\tau^{-1}(\{\mathbf{r}\})\}) = \frac{1}{Z_\Lambda^{O(n)}} \prod_{\gamma\in\Gamma_\Lambda} \frac{n^{r_\gamma}}{r_\gamma!} \mathcal{J}(\gamma)^{r_\gamma} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x(\mathbf{r}))} = \mu_\Lambda^n(\mathbf{r}). \tag{6.48}
$$

The result is thus proved. ☐

6.4 Correlation functions and loops

This section is devoted to a characterisation of certain correlation functions of O(n) spin systems via loop properties. Though this formulation is not straightforward, it turns out to be useful in Chapter 7, where we attempt to understand some properties of loop configurations described by μ_Λ^n. Firstly, we focus on the 2-point correlation function $\langle \varphi_{x_1}^1 \varphi_{x_1}^2 \varphi_{x_2}^1 \varphi_{x_2}^2 \rangle_\Lambda^n$, which can be expressed in terms of loops properties as follows.

Theorem 6.6. *Let μ_Λ^n be the measure over $\mathcal{R}_\Lambda$ as in Eq. (6.34), and let $\mathbb{E}_\Lambda^n[\cdot]$ be* *the expectation with respect to it. Then for any x_1, x_2 distinct sites in Λ*

$$\langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^n_\Lambda = \frac{1}{2n}\mathbb{E}^n_\Lambda\left[\frac{1}{\left(n_{x_1}+\frac{n}{2}\right)\left(n_{x_2}+\frac{n}{2}\right)}\sum_{\substack{\gamma\in\Gamma_\Lambda:\\ x_1,x_2\in\gamma}} r_\gamma\, n_{x_1}(\gamma)\, n_{x_2}(\gamma)\right].$$

A similar description in terms of loops can be proved also for the 4-point correlation function $\langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle^n_\Lambda$.

Theorem 6.7. *Let μ^n_Λ be the measure over $\mathcal{R}_\Lambda$ as in Eq. (6.34), and let $\mathbb{E}^n_\Lambda[\cdot]$ be the expectation with respect to it. Then for any x_1, x_2, x_3, x_4 distinct sites in Λ*

$$\begin{aligned}\langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle^n_\Lambda &= \frac{1}{4n^2}\,\mathbb{E}^n_\Lambda\Bigg[\prod_{i=1}^4 \frac{1}{n_{x_i}+\frac{n}{2}} \\ &\cdot\Bigg(\sum_{p\in\mathscr{P}_4}\sum_{\substack{\gamma,\gamma'\in\Gamma_\Lambda:\gamma\neq\gamma'\\ x_{k_1(p)},x_{m_1(p)}\in\gamma,\\ x_{k_2(p)},x_{m_2(p)}\in\gamma'}} r_\gamma\, r_{\gamma'}\, n_{x_{k_1(p)}}(\gamma)\, n_{x_{m_1(p)}}(\gamma)\, n_{x_{k_2(p)}}(\gamma')\, n_{x_{m_2(p)}}(\gamma') \\ &+ \sum_{\substack{\gamma\in\Gamma_\Lambda:\\ x_1,x_2,x_3,x_4\in\gamma}} f(r_\gamma)\prod_{i=1}^4 n_{x_i}(\gamma) + \frac{n}{2}\, r_\gamma W(\gamma)\binom{W(\gamma)+2}{3}\prod_{i=1}^4 n_{x_i}(\hat{\gamma})\Bigg)\Bigg],\end{aligned}$$

with

$$f(r_\gamma) = \begin{cases} 0 & \text{if } r_\gamma = 0,1; \\ r_\gamma!! & \text{if } r_\gamma > 1 \text{ and odd;} \\ (r_\gamma - 1)!! & \text{if } r_\gamma \text{ even.}\end{cases}$$

Both these theorems follow from Theorem 6.2. Their proofs are the object of the next two sections.

6.4.1 Proof of Theorem 6.6

Recall that from Theorem 6.2 we have

$$\begin{aligned}Z^{O(n)}_\Lambda \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^n_\Lambda = &\sum_{\substack{\omega_1:x_1\to x_2\\ \omega_2:x_2\to x_1}} \mathcal{J}(\omega_1)\mathcal{J}(\omega_2)\sum_{k\geq 0}\frac{n^k}{k!} \\ &\cdot \sum_{\gamma_1,\ldots,\gamma_k\in\Gamma_\Lambda}\prod_{i=1}^k \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x)},\end{aligned} \tag{6.49}$$

with

$$\mathrm{e}^{-\mathcal{V}(n_z)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\sum_{i=1}^k n_z(\gamma_i) + n_z(\omega_1) + n_z(\omega_2) + \frac{n}{2}\right)} \left(\frac{1}{2}\right)^{\sum_{i=1}^k n_z(\gamma_i) + n_z(\omega_1) + n_z(\omega_2)} . \tag{6.50}$$

It is natural to rearrange this expression so that the two paths together create a unique loop to which x_1 and x_2 belong. We denote this extra loop as γ_0. To be precise, if $\omega_1 = (x_1, z_1, \dots, z_l, x_2)$ and $\omega_2 = (x_2, w_1, \dots, w_m, x_1)$ for some $l, m \in \mathbb{N}$, the loop formed by these two paths together is simply $\gamma_0 = (x_1, z_1, \dots, z_l, x_2, w_1, \dots, w_m)$. Notice that there are $n_{x_1}(\hat{\gamma}_0) n_{x_2}(\hat{\gamma}_0) W(\gamma_0)\, 2^{\mathbb{1}(\ell(\hat{\gamma}_0)>2)}$ distinct pairs of open paths joining x_1 and x_2 that produce the same loop γ_0. Fixed arbitrarily an orientation of the loop (clockwise or counterclockwise), the factor $n_{x_1}(\hat{\gamma}_0) n_{x_2}(\hat{\gamma}_0)$ counts the number of possible ways each path can take when leaving their starting site (a part from the winding). The factor $W(\gamma_0)$ is due to the fact that if the loop winds up on itself, the winding must be distributed amongst the two paths – i.e. the $(W(\gamma_0) - 1)$ extra windings of the loop must be distributed in all possible ways between the two paths, which might go around the whole loop once or more before actually arriving at their destination. This is equivalent to distributing $W(\gamma_0) - 1$ indistinguishable balls between 2 distinguishable boxes – the number of ways of doing so is indeed $W(\gamma_0)$. The factor 2 is due to the fact that, if $\ell(\hat{\gamma}) > 2$, for any pair ω_1, ω_2 creating γ_0 we can find another one by inverting the direction of both paths – i.e. it takes care of the fact that the two paths might create the loop both in its clockwise or in its counterclockwise variant.

Furthermore, notice that $\mathcal{J}(\gamma_0) = \mathcal{J}(\omega_1)\mathcal{J}(\omega_2)$. Moreover, $n_u(\gamma_0) = n_u(\omega_1) + n_u(\omega_2)$ if $u \neq x, y$ and $n_u(\gamma_0) = n_u(\omega_1) + n_u(\omega_2) - 1$ if $u = x, y$. We thus have

$$\begin{aligned} Z_\Lambda^{O(n)} \langle \varphi_{x_1}^1 \varphi_{x_1}^2 \varphi_{x_2}^1 \varphi_{x_2}^2 \rangle_\Lambda^n = \frac{1}{4} \sum_{\substack{\gamma_0 \in \Gamma_\Lambda: \\ x_1, x_2 \in \gamma_0}} \mathcal{J}(\omega_0) W(\gamma_0)\, n_{x_1}(\hat{\gamma}_0)\, n_{x_2}(\hat{\gamma}_0)\, 2^{\mathbb{1}(\ell(\hat{\gamma}_0)>2)} \\ \cdot \sum_{k \geq 0} \frac{n^k}{k!} \sum_{\gamma_1, \dots, \gamma_k \in \Gamma_\Lambda} \prod_{i=1}^k \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma})=2)} \frac{1}{\left(n_{x_1} + \frac{n}{2}\right)\left(n_{x_2} + \frac{n}{2}\right)} \mathrm{e}^{-\sum_{z \in \Lambda} \mathcal{V}(n_z)}, \end{aligned} \tag{6.51}$$

with $n_z = \sum_{i=0}^k n_z(\gamma_i)$ and $\mathcal{V}$ of the form

$$\mathrm{e}^{-\mathcal{V}(n_z)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(n_z + \frac{n}{2}\right)} \left(\frac{1}{2}\right)^{n_z} . \tag{6.52}$$

Here we have used the property of the Gamma function that $\Gamma(a+1) = a\Gamma(a)$. We can now rearrange the sum to find

$$Z_\Lambda^{O(n)}\langle\varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^n_\Lambda = \frac{1}{2n}\sum_{k\geq 1}\frac{n^k}{(k-1)!} \\ \cdot\sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda}\left(\frac{1}{k}\sum_{i=1}^k \mathbb{1}\left(x_1,x_2\in\gamma_i\right)n_{x_1}(\gamma_i)n_{x_2}(\gamma_i)\right) \\ \cdot\prod_{i=1}^k\frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)}\frac{1}{\left(n_{x_1}+\frac{n}{2}\right)\left(n_{x_2}+\frac{n}{2}\right)}\,\mathrm{e}^{-\sum_{z\in\Lambda}\mathcal{V}(n_z)}. \tag{6.53}$$

Here we have used the fact that $2^{\mathbb{1}(\ell(\hat{\gamma})>2)} = 2\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma})=2)}$ and $n_x(\gamma) = W(\gamma)n_x(\hat{\gamma})$. Moreover, we have reorganised the sum so that the loop containing x_1 and x_2 does not play any special role. It is then clear that

$$Z_\Lambda^{O(n)}\langle\varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^n_\Lambda = \frac{1}{2n}\sum_{k\geq 1}\frac{n^k}{k!}\sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda}\left(\sum_{i=1}^k \mathbb{1}\left(x_1,x_2\in\gamma_i\right)n_{x_1}(\gamma_i)n_{x_2}(\gamma_i)\right) \\ \cdot\prod_{i=1}^k\frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma})=2)}\frac{\mathrm{e}^{-\sum_{z\in\Lambda}\mathcal{V}(n_z)}}{\left(n_{x_1}+\frac{n}{2}\right)\left(n_{x_2}+\frac{n}{2}\right)}. \tag{6.54}$$

We can now rewrite the expression above in terms of loop configurations $\mathbf{r}\in\mathcal{R}_\Lambda$. What we find is

$$Z_\Lambda^{O(n)}\langle\varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^n_\Lambda = \frac{1}{2n}\sum_{\mathbf{r}\in\mathcal{R}_\Lambda}\prod_{\gamma\in\Gamma_\Lambda}\frac{n^{r_\gamma}}{r_\gamma!}\left(\frac{\mathcal{J}(\gamma)}{W(\gamma)}\right)^{r_\gamma}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma})=2)r_\gamma} \\ \cdot\left(\sum_{\substack{\gamma\in\Gamma_\Lambda:\\ x_1,x_2\in\gamma}} n_{x_1}(\gamma)\,n_{x_2}(\gamma)\,r_\gamma\right)\frac{\mathrm{e}^{-\sum_{z\in\Lambda}\mathcal{V}(n_z(\mathbf{r}))}}{\left(n_{x_1}(\mathbf{r})+\frac{n}{2}\right)\left(n_{x_2}(\mathbf{r})+\frac{n}{2}\right)}. \tag{6.55}$$

The statement follows by dividing both sides by $Z_\Lambda^{O(n)}$.

6.4.2 Proof of Theorem 6.7

Define

$$\varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4} = \mathcal{A}_{x_1x_2x_3x_4} \tag{6.56}$$

Due to the structure of $\mathcal{A}_{x_1x_2x_3x_4}$, from Theorem 6.2 follows that

$$
\begin{aligned}
Z_\Lambda^{O(n)} \langle \mathcal{A}_{x_1x_2x_3x_4} \rangle_\Lambda^n = & \sum_{p,q\in\mathscr{P}_4} \sum_{\substack{\omega_1 : x_{k_1(p)} \to x_{m_1(p)} \\ \omega_2 : x_{k_1(q)} \to x_{m_1(q)} \\ \omega_3 : x_{k_2(p)} \to x_{m_2(p)} \\ \omega_4 : x_{k_2(q)} \to x_{m_2(q)}}} \prod_{i=1}^{4} \mathcal{J}(\omega_i) \\
& \cdot \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x)},
\end{aligned}
\tag{6.57}
$$

with

$$
\mathrm{e}^{-\mathcal{V}(n_z)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\sum_{i=1}^{k} n_z(\gamma_i) + \sum_{i=1}^{4} n_z(\omega_i) + \frac{n}{2}\right)} \left(\frac{1}{2}\right)^{\sum_{i=1}^{k} n_z(\gamma_i) + \sum_{i=1}^{4} n_z(\omega_i)} . \tag{6.58}
$$

In equation (6.57), all possible quadruplets of paths which are suitable to link the sites $x_1, \dots, x_4$ creating loops appear. Since $|\mathscr{P}_4|^2 = 9$, there are nine different types of quadruplets of paths, depending on which sites each path joins. Of these nine types, three are composed by quadruplets of paths such that two paths are between the same two sites and the other two paths join the other two sites (i.e., these quadruplets of paths are related to $p, q \in \mathscr{P}_4$ such that $p = q$). An example of a such a quadruplet of paths is $\omega_1 : x_1 \to x_2$, $\omega_2 : x_2 \to x_1$, $\omega_3 : x_3 \to x_4$, $\omega_4 : x_4 \to x_3$. The remaining six types of quadruplets of paths are such that they go consequently trough all the four sites. An example is $\omega_1 : x_1 \to x_2$, $\omega_2 : x_2 \to x_3$, $\omega_3 : x_3 \to x_4$, $\omega_4 : x_4 \to x_1$. We thus divide the sum above in two pieces, depending on the type of quadruplets present:

$$
Z_\Lambda^{O(n)} \langle \mathcal{A}_{x_1x_2x_3x_4} \rangle_\Lambda^n = \mathscr{O}_1(x_1, x_2, x_3, x_4) + \mathscr{O}_2(x_1, x_2, x_3, x_4), \tag{6.59}
$$

with $\mathscr{O}_1(x_1, x_2, x_3, x_4)$ containing only the contribution of the quadruplets of paths joining sites in pairs, and $\mathscr{O}_2(x_1, x_2, x_3, x_4)$ with the contribution given by quadruplets of paths joining consequently the four sites. We now examine these two pieces separately.

Analysis of $\mathscr{O}_1(x_1, x_2, x_3, x_4)$

Let us firstly focus on $\mathscr{O}_1(x_1, x_2, x_3, x_4)$. The paths appearing here are related to pairs $p, q \in \mathscr{P}_4$ such that $p = q$. We can thus consider only one element of $\mathscr{P}_4$ and

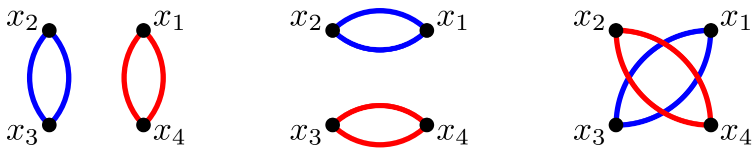

Figure 6.5: Possible ways of linking four sites with two loops.

write:

$$\mathscr{O}_1(x_1,x_2,x_3,x_4) = \sum_{p\in\mathscr{P}_4} \sum_{\substack{\omega_1:x_{k_1(p)}\to x_{m_1(p)}\\ \omega_2:x_{m_1(p)}\to x_{k_1(p)}\\ \omega_3:x_{k_2(p)}\to x_{m_2(p)}\\ \omega_4:x_{m_2(p)}\to x_{k_2(p)}}} \prod_{i=1}^{4} \mathcal{J}(\omega_i)$$
$$\cdot \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\ldots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_{x\in\Lambda}\mathcal{V}(n_x)}, \tag{6.60}$$

with $\mathcal{V}$ as in Eq. (6.58). The quadruplets of loops appearing in $\mathscr{O}_1(x_1,x_2,x_3,x_4)$ join sites in pairs, thus forming two loops, see Fig. 6.5. Let us now define these loops properly. Given $\omega_1 : x_i \to x_j$ and $\omega_2 : x_j \to x_i$ with the form $\omega_1 = (x_i, z_1, \ldots, z_l, x_j)$, $\omega_2 = (x_j, w_1, \ldots, w_m, x_i)$ for some $l, m \in \mathbb{N}$. The composite loop γ_0 is $\gamma_0 = (x_i, z_1, \ldots, z_l, x_j, w_1, \ldots, w_m)$. Notice that for a given γ_0 there are $n_{x_i}(\hat{\gamma}_0) n_{x_j}(\hat{\gamma}_0) W(\gamma_0) 2^{\mathbb{1}(\ell(\hat{\gamma})>2)}$ pairs of paths $\omega_1 : x_i \to x_j$ and $\omega_2 : x_j \to x_j$ that can compose it, as explained in the proof of Theorem 6.6. Moreover, $\mathcal{J}(\gamma_0) = \mathcal{J}(\omega_1)\mathcal{J}(\omega_2)$ and $n_u(\gamma_0) = n_u(\omega_1) + n_u(\omega_2) - \delta_{ux_i} - \delta_{ux_j}$. So, proceeding in the same way as in Theorem 6.6 we find

$$\mathscr{O}_1(x_1,x_2,x_3,x_4) = \frac{1}{4} \sum_{p\in\mathscr{P}_4} \sum_{\substack{\gamma_0,\gamma_0'\in\Gamma_\Lambda:\\ x_{k_1(p)},x_{m_1(p)}\in\gamma_0,\\ x_{k_2(p)},x_{m_2(p)}\in\gamma_0'}} \frac{\mathcal{J}(\gamma_0)\mathcal{J}(\gamma_0')W(\gamma_0)W(\gamma_0')}{2^{\mathbb{1}(\ell(\hat{\gamma}_0)=2)+\mathbb{1}(\ell(\hat{\gamma}_0')=2)}}$$
$$\cdot\, n_{x_{k_1(p)}}(\hat{\gamma}_0)\, n_{x_{m_1(p)}}(\hat{\gamma}_0)\, n_{x_{k_2(p)}}(\hat{\gamma}_0')\, n_{x_{m_2(p)}}(\hat{\gamma}_0')$$
$$\cdot \sum_{k\geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\ldots,\gamma_k\in\Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \prod_{j=1}^{4} \frac{1}{n_{x_j}+\frac{n}{2}}\, \mathrm{e}^{-\sum_{z\in\Lambda} -\mathcal{V}(n_z)}, \tag{6.61}$$

with $n_z = \sum_{i=1}^{k} n_z(\gamma_i) + n_z(\gamma_0) + n_z(\gamma_0')$ and $\mathcal{V}$

$$\mathrm{e}^{-\mathcal{V}(n_z)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(n_z + \frac{n}{2}\right)} \left(\frac{1}{2}\right)^{n_z}. \tag{6.62}$$

We have used $\Gamma(a+1) = a\Gamma(a)$ and $\mathbb{1}(\ell(\hat{\gamma}) = 2) + \mathbb{1}(\ell(\hat{\gamma}) > 2) = 1$. We can now rearrange the sum in order to accommodate γ_0 and γ_0' more naturally.

$$\begin{aligned}\mathscr{O}_1(x_1,x_2,x_3,x_4) = \frac{1}{4}\sum_{k\geq 2}\frac{n^{k-2}}{(k-2)!}\sum_{\gamma_1,\dots,\gamma_k\in\Gamma_\Lambda}\prod_{i=1}^{k}\frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_{z\in\Lambda}-\mathcal{V}(n_z)} \\ \cdot\sum_{p\in\mathscr{P}_4}\left(\frac{1}{k(k-1)}\sum_{i\neq j}\mathbb{1}\left(x_{k_1(p)},x_{m_1(p)}\in\gamma_i\right)\mathbb{1}\left(x_{k_2(p)},x_{m_2(p)}\in\gamma_j\right)\right. \\ \left.\cdot\, n_{x_{k_1(p)}}(\gamma_i)\, n_{x_{m_1(p)}}(\gamma_i)\, n_{x_{k_2(p)}}(\gamma_j)\, n_{x_{m_2(p)}}(\gamma_j)\right)\prod_{j=1}^{4}\frac{1}{n_{x_j}+\frac{n}{2}}.\end{aligned} \tag{6.63}$$

We can reformulate the expression above in terms of configurations $\mathbf{r}\in\mathcal{R}_\Lambda$.

$$\begin{aligned}\mathscr{O}_1(x_1,x_2,x_3,x_4) = \frac{1}{4n^2}\sum_{\mathbf{r}\in\mathcal{R}_\Lambda}\prod_{\gamma\in\Gamma_\Lambda}\frac{n^{r_\gamma}}{r_\gamma!}\left(\frac{\mathcal{J}(\gamma)}{W(\gamma)}\right)^{r_\gamma}\left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)r_\gamma} \mathrm{e}^{-\sum_{z\in\Lambda}\mathcal{V}(n_z)} \\ \cdot\left(\sum_{p\in\mathscr{P}_4}\sum_{\substack{\gamma,\gamma'\in\Gamma_\Lambda:\gamma\neq\gamma' \\ x_{k_1(p)},x_{m_1(p)}\in\gamma, \\ x_{k_2(p)},x_{m_2(p)}\in\gamma'}} r_\gamma\, r_{\gamma'}\, n_{x_{k_1(p)}}(\gamma_i)\, n_{x_{m_1(p)}}(\gamma_i)\, n_{x_{k_2(p)}}(\gamma_j)\, n_{x_{m_2(p)}}(\gamma_j)\right. \\ \left.+\sum_{\substack{\gamma\in\Gamma_\Lambda: \\ x_1,x_2,x_3,x_4\in\gamma}}\prod_{i=1}^{4} n_{x_i}(\gamma) f(r_\gamma)\right)\prod_{j=1}^{4}\frac{1}{n_{x_j}+\frac{n}{2}}.\end{aligned} \tag{6.64}$$

Here $f(r_\gamma) = 0$ if $r_\gamma = 1$, $f(r_\gamma) = r_\gamma!!$ if r_γ is odd and $f(r_\gamma) = (r_\gamma - 1)!!$ if r_γ is even. The term $\sum_{\gamma\in\Gamma_\Lambda}\prod_{i=1}^{4} n_{x_i}(\gamma)f(r_\gamma)$ takes care of the fact that in the second line of Eq. (6.63) the loops γ_i, γ_j considered in the sum might be two different copies of the same loop containing all four sites. $f(r_\gamma)$ counts how many times this could happen with r_γ copies of the same loop. We then have that

$$\begin{aligned}\frac{\mathscr{O}_1(x_1,x_2,x_3,x_4)}{Z_\Lambda^{O(n)}} &= \frac{1}{4n^2}\,\mathbb{E}_\Lambda^n\Bigg[\prod_{i=1}^{4}\frac{1}{n_{x_i}+\frac{n}{2}}\\ &\cdot\Bigg(\sum_{p\in\mathscr{P}_4}\sum_{\substack{\gamma,\gamma'\in\Gamma_\Lambda:\gamma\neq\gamma'\\ x_{k_1(p)},x_{m_1(p)}\in\gamma,\\ x_{k_2(p)},x_{m_2(p)}\in\gamma'}} r_\gamma\, r_{\gamma'}\, n_{x_{k_1(p)}}(\gamma)\, n_{x_{m_1(p)}}(\gamma)\, n_{x_{k_2(p)}}(\gamma')\, n_{x_{m_2(p)}}(\gamma')\\ &+\sum_{\substack{\gamma\in\Gamma_\Lambda:\\ x_1,x_2,x_3,x_4\in\gamma}} f(r_\gamma)\prod_{i=1}^{4} n_{x_i}(\gamma)\Bigg)\Bigg].\end{aligned}\tag{6.65}$$

Analysis of $\mathscr{O}_2(x_1x_2x_3x_4)$

The paths that appear in $\mathscr{O}_2(x_1,x_2,x_3,x_4)$ are those which join the four sites together to create one loop. See Fig. 6.6 for all possible ways of doing this – notice that in $\mathscr{O}_2(x_1,x_2,x_3,x_4)$ each way appears twice, depending on whether the paths follow the loop clockwise or counterclockwise. Instead of examining the paths in $\mathscr{O}_2(x_1,x_2,x_3,x_4)$ in terms of pairs $p,q\in\mathscr{P}_4$ with $p\neq q$, we use the equivalent formulation through permutations of $\{1,2,3,4\}$ with the convention that permutations that differ only by a cyclic transformation are identified (i.e. $(1,2,3,4)=(4,1,2,3)$). Notice that there are 6 such permutations (precisely as the number of types of quadruplets of paths we are considering). With this notation we have

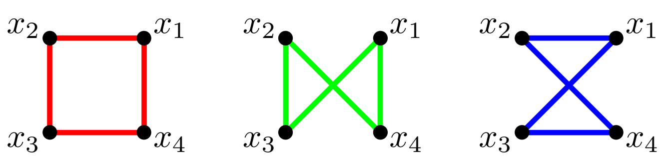

Figure 6.6: Possible ways of linking four sites in a loop, save for the clockwise or counterclockwise orientation.

$$
\begin{aligned}
\mathscr{O}_2(x_1,x_2,x_3,x_4) = &\sum_{\Pi} \sum_{\substack{\omega_1 : x_{\Pi(1)} \to x_{\Pi(2)} \\ \omega_2 : x_{\Pi(2)} \to x_{\Pi(3)} \\ \omega_3 : x_{\Pi(3)} \to x_{\Pi(4)} \\ \omega_4 : x_{\Pi(4)} \to x_{\Pi(1)}}} \prod_{i=1}^{4} \mathcal{J}(\omega_i) \\
&\cdot \sum_{k \geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\ldots,\gamma_k \in \Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_{x\in\Lambda} \mathcal{V}(n_x)},
\end{aligned}
\tag{6.66}
$$

with $\mathcal{V}$ as in Eq. (6.58). Let us now define precisely the loop formed by the four paths. Let Π be a permutation as just described. Let us consider the four paths ω_1, ω_2, ω_3, ω_4 and suppose they have explicit forms $\omega_1 = \big(x_{\Pi(1)}, z_1, \ldots, z_l, x_{\Pi(2)}\big)$, $\omega_2 = \big(x_{\Pi(2)}, v_1, \ldots, v_m, x_{\Pi(3)}\big)$, $\omega_3 = \big(x_{\Pi(3)}, u_1, \ldots, u_t, x_{\Pi(4)}\big)$ and $\omega_4 = \big(x_{\Pi(4)}, w_1, \ldots, w_q, x_{\Pi(1)}\big)$ for some $l, m, t, q \in \mathbb{N}$. The composite loop γ_0 is

$$
\gamma_0 = \big(x_{\Pi(1)}, z_1, \ldots, z_l, x_{\Pi(2)}, v_1, \ldots, v_m, x_{\Pi(3)}, u_1, \ldots, u_t, x_{\Pi(4)}, w_1, \ldots, w_q\big). \tag{6.67}
$$

Notice that $\mathcal{J}(\gamma_0) = \prod_{i=1}^4 \mathcal{J}(\omega_i)$ and $n_u(\gamma_0) = \sum_{i=1}^4 n_u(\omega_i)$ if $u \neq x_1, x_2, x_3, x_4$ and $n_u(\gamma_0) = \sum_{i=1}^4 n_u(\omega_i) - 1$ if $u = x_j$ for some $j \in \{1, \ldots, 4\}$. Moreover, given a certain γ_0 going through four sites in a certain order (apart from the clockwise or counterclockwise orientation), there are $2\,\binom{W(\gamma_0)+2}{3} \prod_{i=1}^4 n_{x_i}(\hat{\gamma}_0)$ quadruples of paths that create that loop. The factor 2 counts the possible orientations (we do not need to worry about the case $\ell(\hat{\gamma}) = 2$ because we are considering distinct sites). The second factor $\prod_{i=1}^4 n_{x_i}(\hat{\gamma}_0)$ counts – fixed an orientation – the number of possible ways each path can take when leaving its starting site (apart from the winding). The binomial factor takes care of the possibility that $W(\gamma_0) > 1$. The argument is similar to the one in the proof of Theorem 6.6: the loop has at least winding 1, and each of the extra $W(\gamma_0) - 1$ windings can be distributed amongst the various paths which constitute it – i.e. each ω_i could wind up some times around the whole final loop before actually arriving to $x_{\Pi(x_{i+1})}$. This is equivalent to sorting $W(\gamma_0) - 1$ indistinguishable balls inside 4 distinguishable boxes, which can be done in $\binom{W(\gamma_0)+2}{3}$ different ways. This implies that

$$
\begin{aligned}
\mathscr{O}_2(x_1,x_2,x_3,x_4) = &\frac{1}{8} \sum_{\gamma_0 \ni x_1,x_2,x_3,x_4} \mathcal{J}(\gamma_0) \binom{W(\gamma_0)+2}{3} \prod_{i=1}^{4} n_{x_i}(\hat{\gamma}_0) \\
\cdot &\sum_{k \geq 0} \frac{n^k}{k!} \sum_{\gamma_1,\ldots,\gamma_k \in \Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \prod_{i=1}^{4} \frac{1}{n_{x_i} + \frac{n}{2}} \mathrm{e}^{-\sum_{z\in\Lambda} \mathcal{V}(n_z)},
\end{aligned}
\tag{6.68}
$$

with $n_z = \sum_{i=0}^{k} n_z(\gamma_i)$ and

$$\mathrm{e}^{-\mathcal{V}(n_z)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(n_z + \frac{n}{2}\right)} \left(\frac{1}{2}\right)^{n_z}. \tag{6.69}$$

We can now reorganise the sum to find

$$\begin{aligned}\mathscr{O}_2(x_1, x_2, x_3, x_4) = \frac{1}{8n} \sum_{k \geq 1} \frac{n^k}{(k-1)!} \sum_{\gamma_1, \dots, \gamma_k \in \Gamma_\Lambda} \prod_{i=1}^{k} \frac{\mathcal{J}(\gamma_i)}{W(\gamma_i)} \left(\frac{1}{2}\right)^{\mathbb{1}(\ell(\hat{\gamma}_i)=2)} \mathrm{e}^{-\sum_z \mathcal{V}(n_z)} \\ \cdot \left(\frac{1}{k} \sum_{i=1}^{k} \mathbb{1}(x_1, x_2, x_3, x_4 \in \gamma_i) W(\gamma_i) \binom{W(\gamma_0)+2}{3} \prod_{j=1}^{4} n_{x_j}(\hat{\gamma}_i)\right) \prod_{i=1}^{4} \frac{1}{n_{x_i} + \frac{n}{2}}.\end{aligned} \tag{6.70}$$

We can then reformulate $\mathscr{O}_2(x_1, x_2, x_3, x_4)$ in terms of loop configurations $\mathbf{r} \in \mathcal{R}_\Lambda$:

$$\begin{aligned}\mathscr{O}_2(x_1, x_2, x_3, x_4) = \frac{1}{8n} \sum_{\mathbf{r} \in \mathcal{R}_\Lambda} \prod_{\gamma \in \Gamma_\Lambda} \frac{n^{r_\gamma}}{r_\gamma!} \left(\frac{\mathcal{J}(\gamma)}{W(\gamma)}\right)^{r_\gamma} \left(\frac{1}{2}\right)^{r_\gamma \mathbb{1}(\ell(\hat{\gamma})=2)} \mathrm{e}^{-\sum_z \mathcal{V}(n_z(\mathbf{r}))} \\ \cdot \left(\sum_{\substack{\gamma \in \Gamma_\Lambda: \\ x_1, x_2, x_3, x_4 \in \gamma}} r_\gamma W(\gamma) \binom{W(\gamma)+2}{3} \prod_{i=1}^{4} n_{x_i}(\hat{\gamma})\right) \prod_{i=1}^{4} \frac{1}{n_{x_i}(\mathbf{r}) + \frac{n}{2}}.\end{aligned} \tag{6.71}$$

By dividing both sides by $Z_\Lambda^{O(n)}$ we get

$$\frac{\mathscr{O}_2(x_1, x_2, x_3, x_4)}{Z_\Lambda^{O(n)}} = \frac{1}{8n} \mathbb{E}_\Lambda^n \left[\prod_{i=1}^{4} \frac{1}{n_{x_i} + \frac{n}{2}} \sum_{\substack{\gamma \in \Gamma_\Lambda: \\ x_1, x_2, x_3, x_4}} r_\gamma W(\gamma) \binom{W(\gamma)+2}{3} \prod_{i=1}^{4} n_{x_i}(\hat{\gamma})\right]. \tag{6.72}$$

The statement of the theorem thus follows from Eq.s (6.65) and (6.72).

Chapter 7

Random loop models and Poisson-Dirichlet distributions

This chapter is devoted to some conjectures regarding the structure of loops in the loop models described in Chapter 6. These are only examples of a wide zoology of models of interacting loops on a lattice, the so called *loop soup models*, see [78] for a recent review. Some examples of great interest in the probability literature are the random interchange model and the random permutation model [71, 35]. In mathematical physics many loop models have emerged in relation not only with classical spin systems as seen previously, but also with quantum ones [77, 4, 81].

Though the definition of loops might vary from model to model, all these different systems are expected to have some common features. In particular, it has been recently proved by Schramm [71] that the random interachange model on the complete graph undergoes a phase transition such that macroscopic loops appear, and their lengths are distributed according to a PD(1) distribution, a member of the family of Poisson-Dirichlet distributions discussed in Appendix D. The proof of this statement is highly non trivial – nonetheless, it has been proposed that such a behaviour should be shown also by the other loop soup models [33, 78]. The conjecture would be that these models on lattices with dimension at least three should exhibit macroscopic loops. The lengths of these loops should be distributed according to some PD(ϑ) – the value of ϑ depending on the properties of the specific model. In some cases, the fraction of the total volume occupied by macroscopic loops should be related to some interesting physical quantity (for example the magnetisation) [81, 78]. Analytical and numerical support for these conjectures has been provided for some specific models [35, 63, 6, 81].

This chapter is devoted to providing some preliminary heuristic calculations to support these conjectures for loop O(n) models. In the next section we describe the expected scenario for the behaviour of loops and formulate precisely some conjectures regarding μ_Λ^n and ν_Λ^n. The following sections aim to show some evidence that these conjectures should indeed be true. First, we define a stochastic process which is an effective split-merge process for loops – this allows us to estimate the value of ϑ, the parameter identifying the PD(ϑ) distribution according to which the lengths of macroscopic loops should be distributed. We then take a different approach and analyse the correlation functions described in Section 6.4 for the O(2) spin system. We show that the explicit expressions in terms of loops we found in Theorems 6.6 and 6.7 are consistent with Poisson-Dirichlet distribution of the lengths of macroscopic loops and provide us with another way of estimating the value of ϑ. The results and calculations (exact and heuristic) presented in this chapter are part of a work in progress yet to be published.

7.1 Classification of loops and some conjectures

Let us consider some generic loop model on the lattice $(\Lambda, \mathcal{E})$ and let us denote loops by γ. Let us consider a configuration with k loops. We label them so that $\ell(\gamma_1) \geq \ell(\gamma_2) \geq \cdots \geq \ell(\gamma_k)$, where $\ell(\gamma)$ is a suitably defined length for any loop. We define $L_{tot} = \sum_{i=1}^k \ell(\gamma_i)$ – it is expected that there should be some constant $\rho > 0$ such that $L_{tot} \sim \rho|\Lambda|$. Then the following is a random partition of the interval $[0, 1]$:

$$\left(\frac{\ell(\gamma_1)}{L_{tot}}, \ldots, \frac{\ell(\gamma_k)}{L_{tot}} \right). \tag{7.1}$$

Loops can be of three types, depending on their length:

1. Macroscopic: $\ell(\gamma) \sim L_{tot}$, i.e. the fraction of the total length they occupy does not vanish in the limit $L_{tot} \to \infty$.
2. Microscopic: $\ell(\gamma) \sim 1$.
3. Mesoscopic: $1 \ll \ell(\gamma) \ll L_{tot}$.

It is conjectured that in dimension at least three, macroscopic loops appear in the thermodynamic limit and these loops occupy a fraction m of L_{tot} for some $m \in [0, 1]$. The other loops are expected to be microscopic – mesoscopic loops are conjectured to be neglegible. The lengths of macroscopic loops are expected to be distributed according to a PD(ϑ) distribution. Figure 7.1 clarifies this.

In the case of the random loop models described in the previous chapter, we can formulate these conjectures as follows, as per [78].

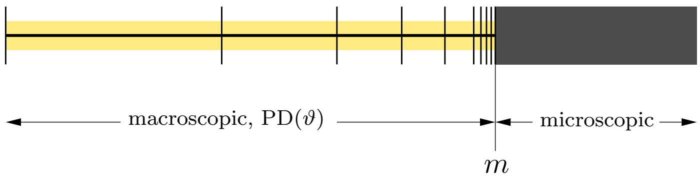

Figure 7.1: Normalised loop lengths as a partition of the interval $[0,1]$. Picture from [78]. m is the fraction of macroscopic loops in the infinite volume limit. They are supposed to distribute according to PD(ϑ) for some $\vartheta > 0$. The fraction $[m,1]$ is occupied by microscopic loops.

Conjecture 7.1. *Let us consider the O(n) loop model described by finite volume measures μ_Λ^n and ν_Λ^n in dimensions at least three. Then there exists $m \in [0,1]$ such that for any $\epsilon > 0$*

$$\lim_{k\to\infty} \lim_{\Lambda\nearrow\mathbb{Z}^d} \mathbb{P}_\Lambda^n \left[\sum_{i=1}^{k} \frac{\ell(\gamma_i)}{L_{tot}} \in [m-\epsilon, m+\epsilon] \right] = 1,$$

$$\lim_{C\to\infty} \lim_{\Lambda\nearrow\mathbb{Z}^d} \mathbb{P}_\Lambda^n \left[\sum_{\substack{i\,\text{s.t.}\\ \ell(\gamma_i)<C}} \frac{\ell(\gamma_i)}{L_{tot}} \in [1-m-\epsilon, 1-m+\epsilon] \right] = 1.$$

with $\mathbb{P}_\Lambda^n$ the relevant (finite volume) probability over loop configurations.

Conjecture 7.2. *Let us consider the O(n) loop model described by the finite volume measures μ_Λ^n and ν_Λ^n, in dimensions at least three. Let $m \in [0,1]$ from Conjecture 7.1. Then there exists $\vartheta > 0$ such that in the limit $L_{tot} \to \infty$, for any $j \in \mathbb{N}$, $\left(\frac{\ell(\gamma_1)}{mL_{tot}}, \dots, \frac{\ell(\gamma_j)}{mL_{tot}}\right)$ converges to the joint distribution of the first j elements of a random partition with PD(ϑ) distribution.*

These conjectures should actually hold for a wide class of loop soup models. In the next sections we explore what would be the right value of ϑ for the O(n) loop models described in the previous chapter. In particular, we expect the following conjecture to be true.

Conjecture 7.3. *For the loop model described by the finite volume measures μ_Λ^n and ν_Λ^n Conjecture 7.2 holds with $\vartheta = \frac{n}{2}$.*

We show in the next section some heuristic calculations in support of this conjecture.

7.2 An effective split-merge process for O(n) loop models

As mentioned in Appendix D, Poisson-Dirichlet distributions are stationary measures of split-merge processes. In particular, a split merge process with parameters g_s and g_m has as invariant measure the distribution $\mathrm{PD}\left(\frac{g_s}{g_m}\right)$. The aim of this section is to define a Markov process with $\nu_\Lambda^n(\bar{m}, \pi)$ as stationary measure and to convince the reader that this is indeed an effective split-merge process on macroscopic loops with $\frac{g_s}{g_m} = \frac{n}{2}$, as hypothesised in Conjecture 7.3. A similar argument has been proposed for loop models related to quantum spin systems in [33, 81, 79, 78].

Recall that for any current configuration $(\bar{m}, \pi) \in \mathcal{C}_\Lambda$, the measure ν_Λ^n is defined as follows:

$$\nu_\Lambda^n(\bar{m}, \pi) = \frac{1}{\mathcal{Z}_\Lambda^n} n^{\mathcal{L}(\bar{m},\pi)} \, \mathrm{e}^{-\sum_{x\in\Lambda} \mathcal{V}\left(\frac{\mathcal{N}_x}{2}\right)} \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!}, \tag{7.2}$$

with $\mathcal{V}$ as per Def. 6.15. We now introduce a stochastic process on the pairings (and the pairings only!) which has this measure as stationary measure. Fixed $\bar{m} \in \mathbb{N}^{\mathcal{E}}$ with $\partial\bar{m} = \emptyset$, the detailed balance condition is

$$\nu_\Lambda^n(\bar{m}, \pi) R(\pi, \pi') = \nu_\Lambda^n(\bar{m}, \pi') R(\pi', \pi). \tag{7.3}$$

with $R(\pi, \pi')$ denoting the rate at which π' occurs when the pairing is π. A process that naturally satisfies this equation is the following:

1. Choose uniformly a site.
2. A different pairing at that site is chosen with rate $\sqrt{n}$ if the change causes a loop to split, thus increasing the total number of loops by one.
3. A different pairing at that site is chosen with rate $\frac{1}{\sqrt{n}}$ if the change merges two loops, thus decreasing the total number of loops by one.
4. A different pairing at that site is chosen with rate 1 if it leaves the total number of loops unvaried.

Let us now argue that if there are macroscopic loops, this process is an effective split-merge for them. Since macroscopic loops would spread all over the lattice, they should interact amongst themselves in an essentially mean-field way. This has a straightforward implication in terms of the heuristic split-merge picture we are describing and its related $\mathrm{PD}(\vartheta)$ distribution. Indeed, let γ_1, γ_2 be two macroscopic

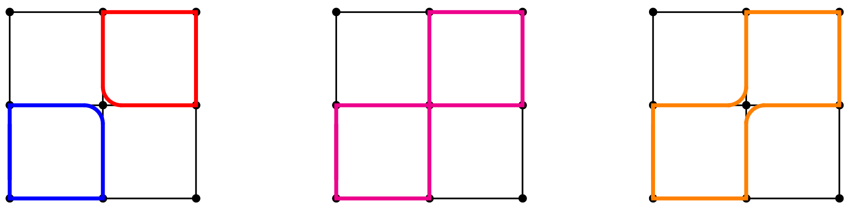

Figure 7.2: Different pairings for the same link configuration. Notice that in the figure on the left there are two loops, while in the others there is one loop only.

loops appearing in the loop configuration $\mathbf{r} = \tau(\bar{m}, \pi)$, with $L_{tot} = \sum_\gamma r_\gamma \ell(\gamma)$. Due to their presumably mean field - like behaviour, there should be a constant k such that each loop gets in contact with itself $\frac{k}{2}\ell(\gamma_i)^2$ times, and the two macroscopic loops get in contact $k\ell(\gamma_1)\ell(\gamma_2)$ times. This means that a macroscopic loop would split at rate $\frac{1}{2}\sqrt{n}\frac{k}{2}\frac{\ell(\gamma_i)^2}{L_{tot}}$ and γ_1, γ_2 would merge together at a rate $\frac{k}{\sqrt{n}}\frac{\ell(\gamma_1)\ell(\gamma_2)}{L_{tot}}$. Notice the extra factor $\frac{1}{2}$ in the splitting case – if a loop intersects with itself at some site, only half of the possible changes in pairing splits it, while the other half only changes its local shape and leaves its length invariant, see Figure 7.2. We can thus define two parameters g_s and g_m such that

$$\begin{aligned} \frac{\sqrt{n}}{2}\frac{k}{2}\frac{\ell(\gamma_1)^2}{L_{tot}} &= g_s\,\ell(\gamma_1)^2, \\ \frac{k}{\sqrt{n}}\frac{\ell(\gamma_1)\ell(\gamma_2)}{L_{tot}} &= 2g_m\ell(\gamma_1)\ell(\gamma_2)\,. \end{aligned} \tag{7.4}$$

This is precisely the situation described in Appendix D for continuous time split-merge process for random partitions (up to a normalising factor for the rates g_s and g_m). Therefore, we can see that the process defined above is an effective split-merge process for macroscopic loops. Its invariant measure is the Poisson-Dirichlet distribution with parameter $\vartheta = \frac{g_s}{g_m} = \frac{n}{2}$. Though of course these calculations are quite far from being formally correct, this heuristic reasoning gives us strong ground to support Conjecture 7.3.

We now examine correlation functions for O(2) models, exploiting Theorem 6.2, in order to provide more evidence in favour of Conjecture 7.3.

7.3 Correlation functions for O(2) models and PD(ϑ) distribution: some heuristics

We have seen in Theorems 6.6 and 6.7 how it is possible to express some correlation functions of O(n) spin models in terms of expectations of some suitable function over loop configurations with respect to the measure μ_Λ^n defined in Eq. (6.34). Though this formulation is not straightforward, it provides us with a way of estimating which PD(ϑ) the lengths of macroscopic loops should be distributed according to.

Remark. In this section we limit ourselves to the O(2) model (also known as XY model). Recall that for this model, Conjectures 7.2 and 7.3 hypothesise that in dimension 3 and higher the lengths of extended loops distribute according to PD(1) distribution. Since for the rest of this section $n = 2$, we drop any dependency on n in the notation. We thus denote by $\langle\cdot\rangle_\Lambda$ the Gibbs state for the XY model, and with μ_Λ, $\mathbb{E}_\Lambda[\cdot]$ the related loop measure and its expectation. Moreover, for the rest of the section we assume that the interaction of the model is translation invariant, i.e. the coupling constants for the O(n) model in Eq. (6.2) assume the same value, $J_{xy} = J_{yx} = J \geq 0$ for any $(x, y) \in \mathcal{E}$. We also fix $\beta = 1$ and drop any dependency on it in the notation.

Our approach goes through the following steps:

1. We estimate explicitly the following limits:

$$\lim_{d(x_1,x_2)\to\infty} \quad \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle; \tag{7.5}$$

$$\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \quad \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle \tag{7.6}$$

   where $\langle\cdot\rangle = \lim_{\Lambda\nearrow\mathbb{Z}^d}\langle\cdot\rangle_\Lambda$. Notice that in the limits above the sites involved in the correlation functions become infinitely far apart one from the other. The main tools used are extremal states decomposition (see Theorem 2.1 and Ex. 2.8) and cluster properties of extremal states (see Theorem 2.2). This step is described in Section 7.3.1

2. We assume Conjecture 7.2 true and using Theorems 6.6 and 6.7 we evaluate the expressions in Eq.s (7.5), (7.6) as functions of ϑ – this passage is the one which is mainly heuristic and where we make many assumptions on the expected behaviour of loops. Moreover, we use extensively some properties of PD(ϑ) distributions discussed in Appendix D. This step is discussed in Section 7.3.2.

3. We show that the two previous steps are consistent only if $\vartheta = 1$. This is the object of Section 7.3.3.

The next sections are devoted to a description of these three steps.

7.3.1 Evaluation of correlation functions via extremal states decomposition

Let us first focus on the 2-point correlation functions. It can be reformulated as follows.

Lemma 7.1. *Let $\langle\cdot\rangle$ be the infinite volume Gibbs state with free boundary conditions of the classical XY model. Then*

$$\lim_{d(x_1,x_2)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle = \frac{\mathcal{F}^2}{2},$$

where $\mathcal{F} = \langle \big(\varphi^1_x\big)^2 - \frac{1}{2}\rangle^0$ for any $x \in \mathbb{Z}^d$.

Proof. Recall from Theorem 2.1 and Ex. 2.8, that for any local function f we have

$$\langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, \langle f\rangle^\alpha \tag{7.7}$$

where $\langle\cdot\rangle = \lim_{\Lambda\nearrow\mathbb{Z}^d}\langle\cdot\rangle_\Lambda$ is the infinite volume state with free boundary conditions and $\langle\cdot\rangle^\alpha = \lim_{\Lambda\nearrow\mathbb{Z}^d}\langle\cdot\rangle^\alpha_\Lambda$ is the extremal state with uniform boundary condition $(\cos\alpha, \sin\alpha)$. Moreover, we can use cluster properties of extremal states from Theorem 2.1 and find:

$$\begin{aligned}
\lim_{d(x_1,x_2)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle &= \frac{1}{2\pi}\lim_{d(x_1,x_2)\to\infty}\int_0^{2\pi} d\alpha \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\alpha \lim_{d(x_1,x_2)\to\infty}\langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle^\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\, \langle \varphi^1_{x_0}\varphi^2_{x_0}\rangle^\alpha \langle \varphi^1_{x_0}\varphi^2_{x_0}\rangle^\alpha,
\end{aligned} \tag{7.8}$$

where x_0 is any arbitrarily chosen site in $\mathbb{Z}^d$. Let us consider $\langle \varphi^1_{x_0}\varphi^2_{x_0}\rangle^\alpha$. Recall that for any $x \in \mathbb{Z}^d$, $\vec{\varphi}_x = (\varphi^1_x,\, \varphi^2_x) = (\cos\varphi_x, \sin\varphi_x)$ with $\varphi_x \in [0, 2\pi)$, so:

$$\langle \varphi^1_x\varphi^2_x\rangle^\alpha = \langle \cos\varphi_x \sin\varphi_x\rangle^\alpha. \tag{7.9}$$

By the rotation invariance of the model and by the properties of sine and cosine we have:

$$\begin{aligned}\langle\varphi_x^1\varphi_x^2\rangle^\alpha &= \langle\cos(\varphi_x-\alpha)\sin(\varphi_x-\alpha)\rangle^0\\ &= (\cos^2\alpha-\sin^2\alpha)\langle\sin\varphi_x\cos\varphi_x\rangle^0+\sin\alpha\cos\alpha\langle\sin^2\varphi_x-\cos^2\varphi_x\rangle^0.\end{aligned} \tag{7.10}$$

Notice $\langle\cdot\rangle^0$ enjoys a $\mathbb{Z}_2$ symmetry, because it is invariant under the following flip of all spins: $\varphi_x^2\to-\varphi_x^2$ for all $x\in\mathbb{Z}^d$. This implies that $\langle\sin\varphi_x\cos\varphi_x\rangle^0 = -\langle\sin\varphi_x\cos\varphi_x\rangle^0=0$. Thus we have

$$\begin{aligned}\langle\varphi_x^1\varphi_x^2\rangle^\alpha &= \sin\alpha\cos\alpha\langle\sin^2\varphi_x-\cos^2\varphi_x\rangle^0\\ &= \sin 2\alpha\left\langle\frac{1}{2}-\cos^2\varphi_x\right\rangle^0\\ &= -\mathcal{F}\sin 2\alpha.\end{aligned} \tag{7.11}$$

This equality and Eq. (7.8) imply

$$\lim_{d(x_1,x_2)\to\infty}\langle\varphi_{x_1}^1\varphi_{x_1}^2\varphi_{x_2}^1\varphi_{x_2}^2\rangle=\frac{\mathcal{F}^2}{2\pi}\int_0^{2\pi}d\alpha\sin^2 2\alpha=\frac{\mathcal{F}^2}{2}. \tag{7.12}$$

□

The same sort of result can be proved for the 4-point correlation function.

Lemma 7.2. *Let $\langle\cdot\rangle$ be the infinite volume Gibbs state with free boundary conditions of the classical XY model. Then*

$$\lim_{\min_{i,j\in\{1,2,3,4\}}d(x_i,x_j)\to\infty}\langle\varphi_{x_1}^1\varphi_{x_1}^2\varphi_{x_2}^1\varphi_{x_2}^2\varphi_{x_3}^1\varphi_{x_3}^2\varphi_{x_4}^1\varphi_{x_4}^2\rangle=\frac{3\mathcal{F}^4}{8},$$

where $\mathcal{F}=\langle\left(\varphi_x^1\right)^2-\frac{1}{2}\rangle^0$ for any $x\in\mathbb{Z}^d$.

Proof. The proof is very similar to the one of Lemma 7.1. Define $\mathcal{A}_x=\varphi_x^1\varphi_x^2$, so $\varphi_{x_1}^1\varphi_{x_1}^2\varphi_{x_2}^1\varphi_{x_2}^2\varphi_{x_3}^1\varphi_{x_3}^2\varphi_{x_4}^1\varphi_{x_4}^2=\mathcal{A}_{x_1}\mathcal{A}_{x_2}\mathcal{A}_{x_3}\mathcal{A}_{x_4}$. By the extremal states decomposition (Theorem 2.1 and Ex. 2.8) and the cluster properties of extremal states we have

$$
\begin{aligned}
&\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \langle \mathcal{A}_{x_1}\mathcal{A}_{x_2}\mathcal{A}_{x_3}\mathcal{A}_{x_4}\rangle \\
&\qquad = \frac{1}{2\pi} \lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \int_0^{2\pi} d\alpha \, \langle \mathcal{A}_{x_1}\mathcal{A}_{x_2}\mathcal{A}_{x_3}\mathcal{A}_{x_4}\rangle^\alpha \\
&\qquad = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \langle \mathcal{A}_{x_1}\mathcal{A}_{x_2}\mathcal{A}_{x_3}\mathcal{A}_{x_4}\rangle^\alpha \\
&\qquad = \frac{1}{2\pi} \int_0^{2\pi} \left(\langle \mathcal{A}_{x_0}\rangle^\alpha\right)^4 .
\end{aligned}
\tag{7.13}
$$

We have seen in Eq. (7.11) that

$$
\langle \mathcal{A}_{x_0}\rangle^\alpha = -\mathcal{F}\sin 2\alpha. \tag{7.14}
$$

This implies that

$$
\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \langle \mathcal{A}_{x_1}\mathcal{A}_{x_2}\mathcal{A}_{x_3}\mathcal{A}_{x_4}\rangle = \frac{\mathcal{F}^4}{2\pi}\int_0^{2\pi} d\alpha \sin^4 2\alpha = \frac{3\mathcal{F}^4}{8}. \tag{7.15}
$$

The statement is thus proved. □

We have thus reformulated our 2-point and 4-point correlation functions with sites at infinite distance in terms of the same parameter $\mathcal{F}$.

7.3.2 Heuristic estimate of correlation functions in terms of ϑ

For the rest of this Chapter we assume that Conjecture 7.2 is true – we have seen in Section 7.2 that we have good reason to believe so. At this stage, we do not assume anything on the value of ϑ. We conjecture the following dependency on the 2-point correlation function on ϑ.

Conjecture 7.4. *Assume $d \geq 3$. Let $\langle\cdot\rangle$ be the infinite volume Gibbs state with free boundary conditions for the classical XY model. Assume Conjecture 7.2 is true. Let $\vartheta > 0$ identify the Poisson-Dirichlet distribution according to which the lengths of macroscopic loops are distributed, and let $m \in [0,1]$ be the fraction of the total loop length occupied by macroscopic loops. Then*

$$
\lim_{d(x_1,x_2)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle = \frac{m^2}{4}\frac{1}{\vartheta+1}.
$$

We now propose an argument in favour of this conjecture, using assumptions and approximations. Let us define $\mathbb{E}[\cdot] = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathbb{E}_\Lambda[\cdot]$ the infinite volume loop expectation. If Conjecture 7.2 is true, this should be equivalent to a $\mathrm{PD}(\vartheta)$ distribution over the lengths of macroscopic loops. We denote by $\mathbb{P}_{\mathrm{PD}(\vartheta)}$ and $\mathbb{E}_{\mathrm{PD}(\vartheta)}[\cdot]$ the probability and expectation for random partitions with Poisson-Dirichlet distribution. We now make the following assumptions:

- The typical macroscopic loop is distributed all over the lattice.
- For the typical macroscopic loop, $W(\gamma) = 1$ and $r_\gamma = 0, 1$. Indeed it seems improbable such a long loop would retrace precisely its steps more than once, as it seems improbable that the same long loop would appear more than once.

Recall from Theorem 6.6 that

$$\langle \varphi^1_{x_1} \varphi^2_{x_1} \varphi^1_{x_2} \varphi^2_{x_2} \rangle_\Lambda = \frac{1}{4} \mathbb{E}_\Lambda \left[\frac{1}{(n_{x_1}+1)(n_{x_2}+1)} \sum_{\substack{\gamma \in \Gamma_\Lambda: \\ x_1, x_2 \in \gamma}} r_\gamma n_{x_1}(\gamma)\, n_{x_2}(\gamma) \right]. \tag{7.16}$$

Then we have

$$\begin{aligned} &\lim_{d(x_1,x_2)\to\infty} \langle \varphi^1_{x_1} \varphi^2_{x_1} \varphi^1_{x_2} \varphi^2_{x_2} \rangle \\ &\qquad = \frac{1}{4} \lim_{d(x_1,x_2)\to\infty} \mathbb{E} \left[\frac{1}{(n_{x_1}+1)(n_{x_2}+1)} \sum_{\gamma : x_1, x_2 \in \gamma} n_{x_1}(\gamma)\, n_{x_2}(\gamma) \right]. \end{aligned} \tag{7.17}$$

In the limit $d(x_1, x_2) \to \infty$, any loop containing both sites must be macroscopic, thus the loops appearing in the sum above are necessarily only macroscopic ones. Notice that there might be more than one loop containing both x_1 and x_2 and the sum above counts how many times the two sites appear together in the same (macroscopic) loop. Moreover, we have that any site x appears $n_x(\mathbf{r})$ times in any loop configuration $\mathbf{r}$, and $n_x(\gamma)$ times in a given loop γ. In terms of Poisson-Dirichlet distribution, this means that different copies of the same site appear in various elements of the partition. Thus the terms $\frac{n_{x_i}(\gamma)}{n_{x_i}+1}$ appear to be normalisations that take care of this fact. These arguments lead us to hypothesise

$$\begin{aligned}
\lim_{d(x_1,x_2)\to\infty} \mathbb{E}\left[\frac{1}{(n_{x_1}+1)(n_{x_2+1})}\sum_{\gamma:x_1,x_2\in\gamma} r_\gamma n_{x_1}(\gamma)n_{x_2}(\gamma)\right] \\
&\approx m^2\,\mathbb{P}_{U,V}\times\mathbb{P}_{\mathrm{PD}(\vartheta)}\,[U,\,V \text{ belong to the same partition element}] \\
&= m^2\,\mathbb{E}_{\mathrm{PD}(\vartheta)}\left[\sum_i X_i^2\right],
\end{aligned} \tag{7.18}$$

In the second line we have the probability that two independent random variables $(U,\,V)$ uniformly distributed in $[0,1]$ with joint probability $\mathbb{P}_{U,V}$ belong to the same element of a random partition with distribution $\mathrm{PD}(\vartheta)$ (see Eq. (D.3)) – in the loop language, this is equivalent to having two sites belonging to the same loop. The second line follows from the first line as explained in Appendix D (see Eq. (D.3)). Notice the factor m^2: macroscopic loops occupy only a fraction m of the total loop length, so the factor m^2 takes care of the fact that we are considering random partitions of $[0,m]$ and not $[0,1]$. It is known (see Lemma D.1) that

$$\mathbb{E}_{\mathrm{PD}(\vartheta)}\left[\sum_i X_i^2\right] = \frac{1}{\vartheta+1}, \tag{7.19}$$

thus supporting Conjecture 7.4. We formulate a similar conjecture for the 4-point correlation functions.

Conjecture 7.5. *Assume $d \geq 3$. Let $\langle\cdot\rangle$ be the infinite volume Gibbs state with free boundary conditions for the classical XY model. Assume Conjecture 7.2 is true. Let $\vartheta > 0$ identify the Poisson-Dirichlet distribution according to which the lengths of extended loops are distributed, and let $m \in [0,1]$ be the fraction of the total loop length occupied by macroscopic loops. Then*

$$\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle = \frac{3m^4}{16(\vartheta+3)(\vartheta+1)}$$

The argument we propose is very similar to the one for Conjecture 7.4.

Firstly, recall that from Theorem 6.7 follows that:

$$
\begin{aligned}
\langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle &= \frac{1}{16}\mathbb{E}\Bigg[\prod_{i=1}^{4}\frac{1}{n_{x_i}+1} \\
&\quad\cdot\Bigg(\sum_{p\in\mathscr{P}_4}\sum_{\substack{\gamma,\gamma':\,\gamma\neq\gamma' \\ x_{k_1(p)},x_{m_1(p)}\in\gamma, \\ x_{k_2(p)},x_{m_2(p)}\in\gamma'}} r_\gamma\, r_{\gamma'}\, n_{x_{k_1}(p)}(\gamma)\, n_{x_{m_1}(p)}(\gamma)\, n_{x_{k_2}(p)}(\gamma')\, n_{x_{m_2}(p)}(\gamma') \\
&\quad+\sum_{\substack{\gamma: \\ x_1,x_2,x_3,x_4\in\gamma}} f(r_\gamma)\prod_{i=1}^{4}n_{x_i}(\gamma)+r_\gamma W(\gamma)\binom{W(\gamma)+2}{3}\prod_{i=1}^{4}n_{x_i}(\hat{\gamma})\Bigg)\Bigg],
\end{aligned}
\tag{7.20}
$$

with

$$
f(r_\gamma)=\begin{cases} 0 & \text{if } r_\gamma=0,1; \\ r_\gamma!! & \text{if } r_\gamma>1 \text{ and odd;} \\ (r_\gamma-1)!! & \text{if } r_\gamma \text{ even.}\end{cases}
\tag{7.21}
$$

Firstly notice that in the limit where the four sites (x_1, x_2, x_3, x_4) are infinitely far from each other, any loop containing two or more of these sites is necessarily macroscopic. This means that both sums above involve macroscopic loops only. As for Conjecture 7.4 we assume that for macroscopic loops $r_\gamma = 0, 1$ and $W(\gamma) = 1$. This means that we can simplify as follows:

$$
\begin{aligned}
&\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty}\langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle \\
&\approx \lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty}\frac{1}{16}\mathbb{E}\Bigg[\prod_{i=1}^{4}\frac{1}{n_{x_i}+1} \\
&\cdot\Bigg(\sum_{p\in\mathscr{P}_4}\sum_{\substack{\gamma,\gamma':\,\gamma\neq\gamma' \\ x_{k_1(p)},x_{m_1(p)}\in\gamma, \\ x_{k_2(p)},x_{m_2(p)}\in\gamma'}} n_{x_{k_1}(p)}(\gamma)n_{x_{m_1}(p)}(\gamma)n_{x_{k_2}(p)}(\gamma')n_{x_{m_2}(p)}(\gamma') \\
&+\sum_{\gamma\ni x_1,x_2,x_3,x_4}\prod_{i=1}^{4}n_{x_i}(\gamma)\Bigg)\Bigg].
\end{aligned}
\tag{7.22}
$$

The first sum counts how many times pairs of sites made from the four sites appear in two distinct loops. Notice there are $|\mathscr{P}_4| = 3$ ways of assigning pairs of sites to two distinct loops (see Fig. 6.5). The second sum counts how many times the four sites appear together in the same loop. Thus, with the same considerations about the terms $\frac{n_{x_i}(\gamma)}{n_{x_i}+1}$ and about m mentioned for Conjecture 7.4, we can conjecture that

$$
\begin{aligned}
&\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \mathbb{E}\Bigg[\prod_{i=1}^{4}\frac{1}{n_{x_i}+1} \\
&\quad\cdot\Bigg(\sum_{p\in\mathscr{P}_4}\sum_{\substack{\gamma,\gamma':\,\gamma\neq\gamma'\\ x_{k_1(p)},x_{m_1(p)}\in\gamma,\\ x_{k_2(p)},x_{m_2(p)}\in\gamma'}} n_{x_{k_1}(p)}(\gamma)n_{x_{m_1}(p)}(\gamma)n_{x_{k_2}(p)}(\gamma')n_{x_{m_2}(p)}(\gamma')\Bigg)\Bigg] \\
&\quad\approx 3m^4\,\mathbb{P}_{U,V,W,T}\times\mathbb{P}_{\mathrm{PD}(\vartheta)}[U,V \text{ are in the same partition element,} \\
&\qquad W,T \text{ are in a different one}] \\
&\quad= 3m^4\,\mathbb{E}_{\mathrm{PD}(\vartheta)}\left(\sum_{\substack{i,j\geq 1\\ distinct}} X_i^2X_j^2\right).
\end{aligned}
\tag{7.23}
$$

Given four independently uniformly distributed random variables $U,V,W,T\in[0,1]$ with joint probability $\mathbb{P}_{U,V,W,T}$, the second line of the equation above is the probability that two of them fall in a certain partition element and the other two in a different one (in the loop setting, this is equivalent to considering two pairs of sites belonging to two different macroscopic loops), given a random partition with PD(ϑ) distribution. The last line follows from the previous one as explained in Appendix D, Eq. (D.6). Analogously:

$$
\begin{aligned}
&\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \mathbb{E}\left[\sum_{\gamma\ni x_1,x_2,x_3,x_4}\prod_{i=1}^{4}\frac{n_{x_i}(\gamma)}{n_{x_i}+1}\right] \\
&\quad\approx m^4\,\mathbb{P}_{U,V,W,T}\times\mathbb{P}_{\mathrm{PD}(\vartheta)}\left[U,V,W,T \text{ belong to the same partition element}\right] \\
&\quad= m^4\,\mathbb{E}_{\mathrm{PD}(\vartheta)}\left[\sum_i X_i^4\right].
\end{aligned}
\tag{7.24}
$$

In the second line, we see the probability of having four independent uniformly distributed random variables belonging to the same partition element – in the loop scenario, this is equivalent to having four sites belonging to the same loop. The third line follows from the properties of PD(ϑ) distributions in Appendix D, Eq.

(D.6). Conjecture 7.5 would then follow because:

$$\mathbb{E}_{\mathrm{PD}(\vartheta)}\left[\sum_{\substack{i,j\geq 1\\ distinct}} X_i^2 X_j^2\right] = \frac{\vartheta}{(\vartheta+3)(\vartheta+2)(\vartheta+1)}, \tag{7.25}$$

$$\mathbb{E}_{\mathrm{PD}(\vartheta)}\left[\sum_{i} X_i^4\right] = \frac{6}{(\vartheta+3)(\vartheta+2)(\vartheta+1)}, \tag{7.26}$$

due to the explicit form of the moments of PD(ϑ) distributions in Lemma D.1.

7.3.3 Evaluation of ϑ

In this section we link the results from the two previous sections, in order to estimate the value of ϑ. Recall that from Lemmas 7.1 and 7.2 we have

$$\lim_{d(x_1,x_2)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle = \frac{\mathcal{F}^2}{2}, \tag{7.27}$$

$$\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle = \frac{3\mathcal{F}^4}{8}. \tag{7.28}$$

Moreover, if Conjectures 7.4 and 7.5 are true we have

$$\lim_{d(x_1,x_2)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\rangle = \frac{m^2}{4(\vartheta+1)}, \tag{7.29}$$

$$\lim_{\min_{i,j\in\{1,2,3,4\}} d(x_i,x_j)\to\infty} \langle \varphi^1_{x_1}\varphi^2_{x_1}\varphi^1_{x_2}\varphi^2_{x_2}\varphi^1_{x_3}\varphi^2_{x_3}\varphi^1_{x_4}\varphi^2_{x_4}\rangle = \frac{3m^4}{16(\vartheta+3)(\vartheta+1)}. \tag{7.30}$$

These two pairs of equations together imply:

$$\left(\frac{\mathcal{F}}{m}\right)^2 = \frac{1}{2(\vartheta+1)}, \tag{7.31}$$

$$\left(\frac{\mathcal{F}}{m}\right)^4 = \frac{1}{2(\vartheta+3)(\vartheta+1)}. \tag{7.32}$$

In order for these two equation to be consistent we need

$$\left(\frac{1}{2(\vartheta+1)}\right)^2 = \frac{1}{2(\vartheta+3)(\vartheta+1)} \tag{7.33}$$

The only solution to the equation above is $\vartheta = 1$, thus supporting Conjecture 7.3 for the O(2) model. Notice that we also have an estimate on the value of m:

$$m = 2|\mathcal{F}| \tag{7.34}$$

So, via some exact calculation and some heuristics, we have provided some arguments in support of Conjecture 7.3 for the O(2) model. Though much work is needed to make them more solid and more formal, the intuition behind them is clear.

We conclude with a remark on the relationship between the appearance of macroscopic loops and the magnetic phase transition for O(n) models. In this chapter we have fixed $\beta = 1$ – if we had not, we would have found the same results but the $\mathcal{F}$ would have been substituted by a parameter $\mathcal{F}_\beta$ depending on the temperature. Analogously, the fraction m of the total loop length occupied by macroscopic loops would have depended on β. The physical intuition is that macroscopic loops would appear (i.e. $m \neq 0$) when the spin model undergoes its magnetic phase transition. This has been conjectured also for loop models related to quantum spin systems [81, 79, 78], for which it has been suggested that the parameter m should be proportional to the magnetisation of the system.

Chapter 8

Conclusions

In this thesis we have reviewed some recent results in the field of quantum and classical statistical mechanics. We conclude by summarising them together with some ideas for future developments.

In Chapter 4 we investigated the decay of correlations for 2d quantum systems with U(1) symmetry. Mermin-Wagner theorem [56, 28] in its various formulations assures that there is no phase transition for this class of models. The relevant correlation functions are thus expected to decay. We have discussed some recent results from [8] concerning this decay. In particular, we have shown that for a very general class of quantum systems, the relevant correlations must decay at least algebraically fast. In order to prove this result, we use the so called *complex rotation method*. This approach has been used in the literature for specific models [55, 21, 13, 42, 45, 30], but we manage to exploit it in its full generality in a much wider setting. Apparently, bosonic systems do not belong to the class of models described in Chapter 4, so it is still an open challenge to generalise the complex rotation method to these models.

In Chapter 5 we have focused on Griffiths-Ginibre inequalities [32] for quantum and classical XY models. We have reviewed some older results present in the literature, showing that this sort of inequalities holds for the O(2) model. We have also discussed correlation inequalities for the quantum XY model. In the case of spin-$\frac{1}{2}$, correlation inequalities have been proved independently in the literature in various works [31, 67, 9]. In particular we have detailed our proof from [9, 10]. The spin-1 case is less straightforward and indeed we proved Griffiths-Ginibre inequalities for the ground state only [9]. It can be hoped that these correlation inequalities might have a wide range of applications. We have seen how they might help define infinite volume Gibbs state for the quantum XY model [9] and explore the relationship between annealed and quenched averages for XY models with random couplings. A question which remains open is to prove Ginibre inequalities for quantum XY models with higher spin and at any temperature.

Chapters 6 and 7 are devoted to some preliminary results (yet to be published) concerning loop models for $\mathrm{O}(n)$ spin systems. In Chapter 6 we show that the celebrated BFS representation [14] provides us with a measure over loop configurations, and we provide another independent loop formulation for $\mathrm{O}(n)$ models. This is a generalisation of the random current representation [1] for the Ising model. We discuss the relationship between these two representations and show them to be equivalent thanks to a suitably defined mapping τ between them. In Chapter 7 we focus on some conjectures concerning the behaviour of loop models related to $\mathrm{O}(n)$ spin systems. These and many other loop models are expected to exhibit macroscopic loops whose lenghts are distributed according to a Poisson-Dirichlet distribution [71, 81, 33, 78]. Furthermore for $\mathrm{O}(n)$ loop models the Poisson-Dirichlet distribution describing macroscopic loops should be $\mathrm{PD}(\frac{n}{2})$ [63]. We describe a stochastic process which is an effective-split merge process for macroscopic loops and show that this is indeed consistent with the conjectured $\mathrm{PD}\left(\frac{n}{2}\right)$ distribution. Moreover, we analyse some of the spin correlation functions for the classical XY model and via a mix of exact results and heuristic calculations we propose an alternative argument in favour of the $\mathrm{PD}\left(\frac{n}{2}\right)$ conjecture in the case $n = 2$. The results in these two chapters are still in preparation, and there are many possible directions for future research. Concerning the generalised random current representation, it would be interesting to explore which possible applications it might have to the study of $\mathrm{O}(n)$ models. Moreover, much work is still needed to provide more convincing evidence that the conjectures regarding Poisson-Dirichlet distributions for loop $\mathrm{O}(n)$ models indeed hold and to investigate their relationship with the magnetic phase transition of the physical model.

Appendix A

The Trotter formula

In Chapters 4 and 5 we use Trotter formula extensively. This very well known result is often used in quantum mechanics and quantum statistical mechanics, and it can be easily found in the literature. It allows us to decompose the exponential of the sum of two matrices in terms of the product of their exponentials. A version for unbounded operators holds (see for example [36]). This brief appendix is devoted to its simpler version for square matrices, as from [80] Proposition 6.2.

Theorem A.1 (Trotter formula)**.** *For any A and B $n \times n$ matrices, the following holds:*

$$\mathrm{e}^{A+B} = \lim_{m\to\infty} \left(e^{\frac{1}{m}A} e^{\frac{1}{m}B} \right)^m .$$

Proof. Notice that the result is trivial in case A and B commutes, since if so $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$. The interesting case is when $[A, B] \neq 0$. We prove the equivalent statement

$$\mathrm{e}^{A+B} = \lim_{m\to\infty} \left(\left(\mathbb{1} + \frac{1}{m} A \right) \mathrm{e}^{\frac{1}{m}B} \right)^m . \tag{A.1}$$

Notice that

$$\begin{aligned} \mathrm{e}^{A+B} - \left(\mathbb{1} + \frac{1}{m}(A+B) \right)^m &= \sum_{j=0}^{m} \frac{1}{j!}(A+B)^j \left(1 - \frac{m(m-1)\dots(m-j+1)}{m^j} \right) \\ &+ \sum_{j\geq m+1} \frac{1}{j!}(A+B)^j . \end{aligned} \tag{A.2}$$

Observe that the norm of the right hand side of the equation above vanishes in the limit $m \to \infty$, so

$$\mathrm{e}^{A+B} = \lim_{m\to\infty} \left(\mathbb{1} + \frac{1}{m}(A+B) \right)^m . \tag{A.3}$$

For any $m \in \mathbb{N}$ let us now define the matrix C_m as

$$C_m = \left(\mathbb{1} + \frac{1}{m}A\right) \mathrm{e}^{\frac{1}{m}B} - \left(\mathbb{1} + \frac{1}{m}(A+B)\right). \tag{A.4}$$

Notice that $\|C_m\| = O\left(\frac{1}{m^2}\right)$. Moreover by the definition of C_m

$$\begin{aligned}
&\left(\left(\mathbb{1} + \frac{1}{m}A\right) \mathrm{e}^{\frac{1}{m}B}\right)^m - \left(\mathbb{1} + \frac{1}{m}(A+B)\right)^m \\
=& \left(\mathbb{1} + \frac{1}{m}(A+B) + C_m\right)^m - \left(\mathbb{1} + \frac{1}{m}(A+B)\right)^m \\
=& \int_0^1 du \, \frac{d}{du}\left(\mathbb{1} + \frac{1}{m}(A+B) + uC_m\right)^m \\
=& \int_0^1 du \sum_{j=0}^{m-1} \left(\mathbb{1} + \frac{1}{m}(A+B) + uC_m\right)^j C_m \left(\mathbb{1} + \frac{1}{m}(A+B) + uC_m\right)^{m-j-1}.
\end{aligned} \tag{A.5}$$

This implies that

$$\begin{aligned}
&\left\|\left(\left(\mathbb{1} + \frac{1}{m}A\right) \mathrm{e}^{\frac{1}{m}B}\right)^m - \left(\mathbb{1} + \frac{1}{m}(A+B)\right)^m\right\| \\
&\leq m \int_0^1 du \left\|\mathbb{1} + \frac{1}{m}(A+B) + uC_m\right\|^{m-1} \|C_m\| \\
&\leq m \left(1 + \frac{1}{m}\|A+B\| + \|C_m\|\right)^{m-1} \|C_m\|.
\end{aligned} \tag{A.6}$$

Since $\|C_m\| = O\left(\frac{1}{m^2}\right)$, we have that $\lim_{m\to\infty}\left(1 + \frac{1}{m}\|A+B\| + \|C_m\|\right)^{m-1} = \mathrm{e}^{\|A+B\|}$, which implies that the right hand side of the equation above goes to zero. Thus

$$\lim_{m\to\infty} \left(\left(\mathbb{1} + \frac{1}{m}A\right) \mathrm{e}^{\frac{1}{m}B}\right)^m = \lim_{m\to\infty} \left(\mathbb{1} + \frac{1}{m}(A+B)\right)^m. \tag{A.7}$$

By equation (A.3) the result is proved. $\square$

This version of the theorem is a simple one – indeed a more general version for unbounded operators on infinite Hilbert spaces can be proved, see for example [36].

Appendix B

Hölder inequality for matrices

Hölder inequality is a standard result for functions in L^p spaces. Also a matrix version exists, which we use extensively in Chapter 4. This appendix is devoted to a review of the latter formulation and its proof. Before turning to it, we mention the standard result for functions [70].

Theorem B.1. *Let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let X be an arbitrary measure space with positive measure μ. Let $f \in L^p(X)$ and $g \in L^q(X)$. Then $fg \in L^r(X)$ and*

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Here $\|\cdot\|_p$ is the usual p-norm

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}.$$

Notice that this theorem has a straightforward corollary when the measure taken into consideration is the counting measure, which turns out to be useful later on.

Corollary B.1. *Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Then for any p, q, $r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$*

$$\left(\sum_{i=1}^n |x_i y_i|^r \right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}.$$

Let us now turn to matrices. p-norms can be introduced for them as well.

Definition B.1. *Let $n \in \mathbb{N}$ and $p \in [1, \infty]$. For any A $n \times n$ matrix the p-norm is* *defined as*

$$\|A\|_p = \left(\text{tr } (A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}}.$$

This norm can be expressed also in terms of the singular values of the matrix. Recall that for any (square, $n \times n$) matrix A the *singular value decomposition* holds, i.e. there are two unitary (square, $n \times n$) matrices U and V such that

$$A = U^*DV, \tag{B.1}$$

with D a diagonal matrix with non negative elements on the diagonal. These take the name of *singular values* of A, and we denote them by $\lambda_i(A)$, $i \in \{1, \ldots, n\}$. It is customary to label them in decreasing order, i.e. $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \lambda_n(A)$. The following holds.

Proposition B.1. *Let $p \in [1, \infty]$. Let A be an $n \times n$ normal matrix and $\lambda_1(A), \ldots, \lambda_n(A)$ its singular values. Then*

$$\|A\|_p = \left(\sum_{i=1}^{n} \lambda_i(A)^p\right)^{\frac{1}{p}}.$$

Proof. By Eq. (B.1) we have

$$A^*A = V^*D^2V = V^*\text{diag}(\lambda_1(A)^2, \ldots, \lambda_n(A)^2)V. \tag{B.2}$$

By the ciclicity of trace and unitarity of V the result follows. □

It is straightforward to check that $\|\cdot\|_p$ is indeed a norm. Notice that for $p \to \infty$ we recover the usual definition of $\|A\|_\infty$. The p-norm can be equivalently formulated as follows.

We can now formulate Hölder inequality in its matrix formulation.

Theorem B.2 (Hölder inequality for matrices)**.** *Let A and B be $n \times n$ matrices and let $q, p, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then*

$$\|AB\|_r \leq \|A\|_p \|B\|_q.$$

Proof. Let $\lambda_i(A)$, $\lambda_i(B)$ and $\lambda_i(AB)$ with $i \in \{1, \ldots, n\}$ be the singular values of A,

B and AB respectively. It is known (see e.g. [7], Theorem IV.2.5) that

$$\sum_{i=1}^{n} \lambda_i(AB)^r \leq \sum_{i=1}^{n} \lambda_i(A)^r \lambda_i(B)^r \quad \forall r > 0. \tag{B.3}$$

From this and Corollary B.1 we get

$$\begin{aligned} \|AB\|_r &= \left(\sum_{i=1}^{n} \lambda_i(AB)^r \right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} \lambda_i(A)^r \lambda_i(B)^r \right)^{\frac{1}{r}} \\ &\leq \left(\sum_{i=1}^{n} \lambda_i(A)^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \lambda_i(B)^q \right)^{\frac{1}{q}} = \|A\|_p \|B\|_q. \end{aligned} \tag{B.4}$$

The result is thus proved. $\square$

Different proofs can be found in [25, 75].

Appendix C

The random current representation of the Ising model

The goal of this appendix is to briefly review the random current representation for the Ising model. It was first proposed by Aizenman in his seminal paper [1] and successfully exploited in the literature [74, 2, 3]. See [23, 18] for recent reviews.

Let $(\Lambda, \mathcal{E})$ be the lattice, with $\Lambda \subset\subset \mathbb{Z}^d$ the set of vertices and $\mathcal{E}$ the set of edges. On each site of the lattice let there be a discrete spin which can assume only two values: $\sigma_x \in \{-1, 1\}$ $\forall x \in \Lambda$. The energy of a configuration $\sigma = \{\sigma_x\}_{x\in\Lambda}$ is given by the following hamiltonian.

$$H^{\mathrm{Is}}_{\Lambda}(\sigma) = -\frac{1}{2} \sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y, \tag{C.1}$$

with

$$J_{xy} = J_{yx} \begin{cases} \geq 0 \text{ if } (x, y) \in \mathcal{E}; \\ = 0 \text{ otherwise.} \end{cases} \tag{C.2}$$

The definition of the partition function is the usual one,

$$Z^{\mathrm{Is}}_{\Lambda} = \sum_{\sigma\in\{-1,1\}^{\Lambda}} \mathrm{e}^{-H^{\mathrm{Is}}_{\Lambda}(\sigma)}. \tag{C.3}$$

Notice that in the expression above we fix $\beta = 1$ and drop any dependency on it for the rest of the section. Recall the definition of link configuration and sources provided in Section 6.3, Def.s 6.8 and 6.10. The next theorem allows us to reformulate the Ising model in terms of link configurations. Notice that in this context we do not need the notion of pairing introduced in Chapter 6.

Theorem C.1 (Random current representation of the Ising model [1])**.** *Z^{Is}_Λ can be reformulated as follows:*

$$Z^{\mathrm{Is}}_\Lambda = 2^{|\Lambda|} \sum_{\bar{m}:\partial\bar{m}=\emptyset} \prod_{e\in\mathcal{E}} \frac{J_e^{n_e}}{n_e!},$$

where for any $e=(x,y)$, $J_e = J_{xy} = J_{yx}$.

Proof. Notice that the hamiltonian can be reformulated as

$$H^{\mathrm{Is}}_\Lambda(\sigma) = -\sum_{\substack{e\in\mathcal{E}\\ e=(x,y)}} J_e \sigma_x \sigma_y. \tag{C.4}$$

The partition function can be rewritten as

$$\begin{aligned} Z^{\mathrm{Is}}_\Lambda &= \sum_{\sigma\in\{-1,1\}^\Lambda} \prod_{\substack{e\in\mathcal{E}\\ e=(x,y)}} \mathrm{e}^{J_e\sigma_x\sigma_y} \\ &= \sum_{\sigma\in\{-1,1\}^\Lambda} \prod_{\substack{e\in\mathcal{E}\\ e=(x,y)}} \sum_{m_e\geq 0} \frac{J_e^{m_e}}{m_e!} (\sigma_x\sigma_y)^{m_e} \\ &= \sum_{\bar{m}\in\mathbb{N}^\mathcal{E}} \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!} \sum_{\sigma\in\{-1,1\}^\Lambda} \prod_{x\in\Lambda} \sigma_x^{\sum_{e\ni x} m_e} \\ &= \sum_{\bar{m}\in\mathbb{N}^\mathcal{E}} \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!} \prod_{x\in\Lambda} \sum_{\sigma_x=\pm1} \sigma_x^{\sum_{e\ni x} m_e}. \end{aligned} \tag{C.5}$$

Notice that, given $k\in\mathbb{N}$,

$$\sum_{\sigma_x=\pm1} \sigma_x^k = \begin{cases} 0 \text{ if } k \text{ is odd;} \\ 2 \text{ if } k \text{ is even.} \end{cases} \tag{C.6}$$

This implies the statement. □

Notation. Given $\bar{m}$, $\bar{n}\in\mathbb{N}^\mathcal{E}$, $\bar{k}=\bar{m}+\bar{n}$ is defined as the link configuration such that $k_e = m_e + n_e$ for any $e\in\mathcal{E}$.

Remark. We can define a partial ordering over configurations. Let $\bar{m}$, $\bar{n}\in\mathbb{N}^\mathcal{E}$. $\bar{m}\leq\bar{n}$ if for any $e\in\mathcal{E}$ $m_e\leq n_e$.

Given a link configuration, connected sites are defined as follows.

Definition C.1 (Connected sites)**.** *Let $(\Lambda,\mathcal{E})$ be the lattice and $\bar{m}\in\mathbb{N}^\mathcal{E}$ a link configuration. $x,y\in\Lambda$ are connected by $\bar{m}$ if there exists a path $\omega = \{(x,x_1)(x_1,x_2)\ldots$*

$(x_n, y)\}$ with $(u,v) \in \mathcal{E}$ for any $(u,v) \in \omega$ such that $m_e \neq 0$ for all $e \in \gamma$. The notation is $x \overset{\bar{m}}{\longleftrightarrow} y$.

Remark. Notice that $\partial \bar{m} = \{x, y\}$ implies $x \overset{\bar{m}}{\longleftrightarrow} y$ (but not the other way around!).

We conclude this Appendix with a statement concerning link configurations and their sources, the so called Switching Lemma. We propose it here in the formulation from [18].

Lemma C.1 (Switching Lemma)**.** *Let $(G, \mathcal{E}_G) \subset (\Lambda, \mathcal{E})$. Let $x, y \in G$ and $A \subset \Lambda$. Then for any function $f : \mathbb{N}^{\mathcal{E}} \to \mathbb{R}$*

$$\sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\,\partial\bar{m}=\{x,y\}\\ \bar{n}\in\mathbb{N}^{\mathcal{E}}:\,\partial\bar{n}=A}} f(\bar{m}+\bar{n})w(\bar{m})w(\bar{n}) = \sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\,\partial\bar{m}=\emptyset\\ \bar{n}\in\mathbb{N}^{\mathcal{E}}:\,\partial\bar{n}=A\triangle\{x,y\}}} f(\bar{m}+\bar{n})w(\bar{m})w(\bar{n})\mathbb{1}\left(x \overset{\bar{m}+\bar{n}}{\longleftrightarrow} y \text{ in } G\right).$$

Here, $w(\bar{m}) = \prod_{e\in\mathcal{E}} \frac{J_e^{m_e}}{m_e!}$, and $A \triangle B = (A \cup B)\backslash(A \cap B)$ for any sets A, B.

Proof. Throughout the proof each configuration $\bar{m} \in \mathbb{N}^{\mathcal{E}_G}$ is identified with a configuration in $\mathbb{N}^{\mathcal{E}}$ with no links in $\mathcal{E}\backslash\mathcal{E}_G$. We can rearrange the left hand side of the expression above by introducing a new variable $\bar{k} = \bar{m} + \bar{n}$.

$$\sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\,\partial\bar{m}=\{x,y\}\\ \bar{n}\in\mathbb{N}^{\mathcal{E}}:\,\partial\bar{n}=A}} f(\bar{m}+\bar{n})w(\bar{m})w(\bar{n}) = \sum_{\substack{\bar{k}\in\mathbb{N}^{\mathcal{E}}:\\ \partial\bar{k}=A\triangle\{x,y\}}} f(\bar{k})w(\bar{k}) \sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\\ \partial\bar{m}=\{x,y\},\\ \bar{m}\leq\bar{k}}} \binom{\bar{k}}{\bar{m}} \tag{C.7}$$

We have introduced the notation $\binom{\bar{k}}{\bar{m}} = \prod_e \binom{k_e}{m_e}$. The right hand side can be rearranged in a similar way as well:

$$\begin{aligned}
&\sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\,\partial\bar{m}=\emptyset\\ \bar{n}\in\mathbb{N}^{\mathcal{E}}:\,\partial\bar{n}=A\triangle\{x,y\}}} f(\bar{m}+\bar{n})w(\bar{m})w(\bar{n})\mathbb{1}\left(x \overset{\bar{m}+\bar{n}}{\longleftrightarrow} y \text{ in } G\right)\\
&= \sum_{\substack{\bar{k}\in\mathbb{N}^{\mathcal{E}}:\\ \partial\bar{k}=A\triangle\{x,y\}}} f(\bar{k})w(\bar{k})\mathbb{1}\left(x \overset{\bar{k}}{\longleftrightarrow} y \text{ in } G\right) \sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\\ \partial\bar{m}=\emptyset,\\ \bar{m}\leq\bar{k}}} \binom{\bar{k}}{\bar{m}}.
\end{aligned} \tag{C.8}$$

We now need only to prove that the following holds for any $\bar{k} \in \mathbb{N}^{\mathcal{E}}$:

$$\sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\\ \partial\bar{m}=\{x,y\},\\ \bar{m}\leq\bar{k}}} \binom{\bar{k}}{\bar{m}} = \mathbb{1}\left(x \xleftrightarrow{\bar{k}} y \text{ in } G\right) \sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\\ \partial\bar{m}=\emptyset,\\ \bar{m}\leq\bar{k}}} \binom{\bar{k}}{\bar{m}}. \tag{C.9}$$

Firstly, assume that x and y are not connected by $\bar{k}$. Then there is no $\bar{m} \leq \bar{k}$ which connects them, so the left hand side is zero. The right hand side is trivially zero, and the equality holds. Now assume that $\bar{k}$ connects the two sites x and y. Associate to $\bar{k}$ the graph $\mathcal{K}$ which has as vertices the vertices of G and between any pair of sites (u,v) a number of edges equal to $k_{(u,v)}$. For any subgraph $\mathcal{M}$ define $\partial\mathcal{M} = \{u \in \mathcal{M} : u \text{ belongs to an odd number of edges}\}$. Then we have that

$$\begin{aligned} |\{\mathcal{M} \subset \mathcal{K} : \partial\mathcal{M} = \{x,y\}\}| &= \sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\\ \partial\bar{m}=\{x,y\},\\ \bar{m}\leq\bar{k}}} \binom{\bar{k}}{\bar{m}}, & \text{(C.10)} \\ |\{\mathcal{M} \subset \mathcal{K} : \partial\mathcal{M} = \emptyset\}| &= \sum_{\substack{\bar{m}\in\mathbb{N}^{\mathcal{E}_G}:\\ \partial\bar{m}=\emptyset,\\ \bar{m}\leq\bar{k}}} \binom{\bar{k}}{\bar{m}}. & \text{(C.11)} \end{aligned}$$

If x and y are connected by $\bar{k}$, then there exists a subgraph $\mathcal{Y}$ of $\mathcal{K}$ such that $\partial\mathcal{Y} = \{x,y\}$. For any $\mathcal{M}$ with $\partial\mathcal{M} = \emptyset$, notice that $\partial(\mathcal{M} \triangle \mathcal{Y}) = \{x,y\}$. The operation $\mathcal{M} \to \mathcal{M} \triangle \mathcal{Y}$ is a bijection between the sets $\{\mathcal{M} \subset \mathcal{K} : \partial\mathcal{M} = \emptyset\}$ and $\{\mathcal{M} \subset \mathcal{K} : \partial\mathcal{M} = \{x,y\}\}$, which thus have the same size. By Eq.s (C.10) and (C.11), the equality in Eq. (C.9) holds. The statement is thus proved. $\square$

Appendix D

Poisson-Dirichlet distributions

The family of Poisson-Dirichlet distributions, introduced by Kingman [43], is a class of distributions depending on a positive parameter ϑ. It it can be defined in different ways. This appendix is devoted to a brief review of the definition and of some useful properties of this family of distributions. See [19] for a recent review.

One possible way of defining PD(ϑ) distributions is through a *stick breaking construction*. Recall that Beta(ϑ) is the probability distribution on the interval $[0,1]$ with probability density function $\vartheta(1-t)^{\vartheta-1}$ with $t \in [0,1]$. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables distributed according to it. The following is an unordered random partition:

$$(X_1, (1-X_1)X_2, (1-X_1)(1-X_2)X_3, \dots). \tag{D.1}$$

The distribution of such unordered random partitions takes the name of GEM(ϑ) distribution, from the names of Griffiths, Engen and McCloskey. If we reorder this random partition in decreasing order, by definition the new random partition has PD(ϑ) distribution.

Another interesting way to define it is through a so called *split-merge process*. Given two nonnegative parameters g_s, $g_m \in [0,1]$, we define the following discrete-time stochastic process. Let $(Y_1, Y_2, \dots)$ be a partition at a certain time $t \in \mathbb{N}$. The partition at time $t+1$ is found in the following way:

1. Choose with uniform probability two numbers u, v in the interval [0,1].
2. If u and v are in the same interval, with probability g_s split it uniformly.
3. If u and v are in different intervals, with probability g_m merge the two intervals.
4. Rearrange the intervals in decreasing order.

In the equivalent continuous time process, an element Y_i splits at rate $g_s Y_i^2$ and two elements Y_i and Y_j merge at rate $2g_m Y_i Y_j$. It can be shown that the invariant measure for these processes is a Poisson-Dirichlet distribution with parameter $\vartheta = \frac{g_s}{g_m}$, see for example [11, 33, 78] and references therein.

We now review some properties of Poisson-Dirichlet distributions which are used extensively in Chapter 7. We denote by $\mathbb{P}_{\mathrm{PD}(\vartheta)}$ and $\mathbb{E}_{\mathrm{PD}(\vartheta)}$ the probability and expectation for random partitions with Poisson-Dirichlet distribution. Let us take two independent random variables U and V uniformly distributed in the interval [0,1] and let $\mathbb{P}_{U,V}$ denote their joint probability. We would like to calculate the probability that they belong to the same element of a random partition with PD(ϑ) distribution. We denote by $\mathbb{P}_{U,V} \times \mathbb{P}_{\mathrm{PD}(\vartheta)}$ the product measure. Let $\{X_i\}_{i\geq 1}$ denote the elements of the random partition. Firstly notice that the probability that they both belong to a given partition element X_i is

$$\begin{aligned}\mathbb{P}_{U,V} \times \mathbb{P}_{\mathrm{PD}(\vartheta)}\left[U, V \in X_i\right] &= \int_0^1 du \int_0^1 dv\, \mathbb{E}_{\mathrm{PD}(\vartheta)}[\mathbb{1}(u, v \in X_i)] \\ &= \mathbb{E}_{\mathrm{PD}(\vartheta)}[X_i^2].\end{aligned} \tag{D.2}$$

Thus we have

$$\mathbb{P}_{U,V} \times \mathbb{P}_{\mathrm{PD}(\vartheta)}\left[U, V \text{ belong to the same partition element}\right] = \mathbb{E}_{\mathrm{PD}(\vartheta)}\left[\sum_i X_i^2\right]. \tag{D.3}$$

This sort of argument can be generalised straightforwardly. Let $r \in \mathbb{N}$ and let $\{k_i\}_{i=1}^r$ be r positive integers. Let $K = \sum_{i=1}^r k_i$ and define K independent random variables uniformly distributed in the interval $[0, 1]$: $\{U_m^j\}_{1\leq m\leq k_j}^{1\leq j\leq r}$. Let us denote by $\mathbb{P}_{\mathbf{U}}$ their joint probability. For any $j \in \{1, \ldots, r\}$ and a given partition element X_{i_j} we define the following event

$$\mathscr{E}_{i_j}^j = \{U_m^j \in X_{i_j} \quad \forall m \in \{1, \ldots, k_j\}\}. \tag{D.4}$$

This describes the scenario in which all the random variables with upper index j belong to the same given partition element X_{i_j}. We now fix r distinct integers $i_1, \ldots, i_r \geq 1$ and define the following event

$$\mathscr{E}_{i_1,\ldots,i_r} = \bigcap_{j=1}^r \mathscr{E}_{i_j}^r. \tag{D.5}$$

This event describes the scenario in which random variables sharing the same upper index belong to a certain given partition element and the j partition elements thus involved are fixed and distinct.

With a reasoning similar to the one leading to Eq. (D.3) we have that

$$\mathbb{P}_{\mathbf{U}} \times \mathbb{P}_{\mathrm{PD}(\vartheta)} \left[\bigcup_{\substack{i_1,\dots,i_r \geq 1 \\ \text{distinct}}} \mathscr{E}_{i_1,\dots,i_r} \right] = \mathbb{E}_{\mathrm{PD}(\vartheta)} \left[\sum_{\substack{i_1,i_2,\dots,i_j \geq 1 \\ \text{distinct}}} X_{i_1}^{k_1} \dots X_{i_j}^{k_j} \right] . \tag{D.6}$$

The probability appearing here is the probability of having k_1 random variables in the same partition element, k_2 in another one and so on. The moments of Poisson-Dirichlet distributions can be explicitly calculated, as stated in the following Lemma.

Lemma D.1. *Let $j \in \mathbb{N}$ and $k_i \in 2\mathbb{N}$ for any $i \in \{1, \dots, j\}$. Then*

$$\mathbb{E}_{\mathrm{PD}(\vartheta)} \left[\sum_{\substack{i_1,i_2,\dots,i_j \geq 1 \\ \text{distinct}}} X_{i_1}^{k_1} \dots X_{i_j}^{k_j} \right] = \frac{\vartheta^j \Gamma(\vartheta) \Gamma(k_1) \dots \Gamma(k_j)}{\Gamma(\vartheta + k_1 + \dots k_j)}$$

This formula appears in [63, 78], where it is used to explore Conjectures 7.2 and 7.3 for certain loop soup models.

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